INTEGRABILITY OF THE BROUWER DEGREE FOR IRREGULAR ARGUMENTS

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Abstract. We prove that the Brouwer degree $\deg(u, U, \cdot)$ for a function $u \in C^{0,\alpha}(U;\mathbb{R}^n)$ is in $L^p(\mathbb{R}^n)$ if $1 \le p < \frac{n\alpha}{d}$, where $U \subset \mathbb{R}^n$ is open and bounded and d is the box dimension of ∂U. This is supplemented by a theorem showing that $u_j \to u$ in $C^{0,\alpha}(U;\mathbb{R}^n)$ implies $\deg(u_j, U, \cdot) \to \deg(u, U, \cdot)$ in $L^p(\mathbb{R}^n)$ for the parameter regime $1 \le p < \frac{n\alpha}{d}$, while there exist convergent sequences $u_j \to u$ in $C^{0,\alpha}(U;\mathbb{R}^n)$ such that $\|\deg(u_j, U, \cdot)\|_{L^p} \to \infty$ for the opposite regime $p > \frac{n\alpha}{d}$.

1. Introduction

The Brouwer degree is a very useful object in nonlinear analysis, in particular in problems with a geometric background. One notable example of its use is the $C^{1,\alpha}$ isometric immersion problem (see [8]), where the integrability properties of the degree are crucial. For a Lipschitz function $u : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open and bounded, the integrability of the Brouwer degree is as good as one could hope, namely, there is the classical "change of variables"-type formula

$$\int_U \varphi(u(x)) \det Du(x) \mathrm{d}x = \int_{\mathbb{R}^n} \varphi(z) \deg(u, U, z) \mathrm{d}z \tag{1}$$

for all $\varphi \in L^1(\mathbb{R}^n)$ (see e.g. [11]). However, when the regularity of u is worse – only $C^{0,\beta}$ for some $0 < \beta < 1$ – it is much less clear how to deal with integrals as the one on the right hand side above. To obtain information about such integrals, we will use the fact that $\deg(u, U, y)\mathrm{d}y$ is an exact form (see e.g. [12]) and try to apply Stokes' Theorem to write it as a boundary integral. This in turn needs some regularity of the boundary ∂U. Usually, one needs U to be a set of finite perimeter to be able to apply Stokes' Theorem. In [13], it has been shown that if the integrand is smooth enough, then Stokes' Theorem may also be applied to sets with rougher boundary. The first aim of the present paper is to adapt these ideas to the case of the Brouwer degree and show that $\deg(u, U, \cdot)$ is integrable if u is smooth enough in terms of Hölder regularity, and ∂U is smooth enough in terms of its box dimension. We will show that there is a trade-off between these two types of regularity.

Theorem 1.1. *Let $0 < \alpha < 1$ and $n - 1 < d < n$ such that $n\alpha > d$, and let $U \subset \mathbb{R}^n$ be open and bounded with $\dim_{\text{box}} \partial U = d$. Furthermore , let $u \in C^{0,\alpha}(U;\mathbb{R}^n)$. Then $\deg(u, U, \cdot) \in L^p(\mathbb{R}^n)$ for all $1 \le p < \frac{n\alpha}{d}$, and for all $p \in (1, \frac{n\alpha}{d})$, there exists a constant $C = C(n, U, \alpha, d, p)$ such that*

$$\|\deg(u, U, \cdot)\|_{L^p} \le C\|u\|^{n/p}_{C^{0,\alpha}(U;\mathbb{R}^n)} .$$

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In fact, we will prove this theorem by giving a meaning to the left hand side in the change of variables formula (1), with the regularity of u, U as stated in the theorem. We will show how to make sense of the left hand side for $u \in C^{0,\alpha}(U;\mathbb{R}^n)$ and $\varphi \in L^{p'}$ where p' is defined by requiring $p^{-1}+(p')^{-1}=1$. The main idea is to represent $\varphi(u(x))\det Du(x)$ as a sum of Jacobian determinants, interpreted in a weak sense. There are two crucial tools that will allow us to do so. First, we use multi-linear (real) interpolation for a suitable weak definition of the Jacobian determinant, see Lemma 3.1. The statement of this lemma can be viewed as a variant of Theorem 3 in the paper [5] by Brezis and Nguyên, which relies on an idea by Bourgain, Brezis and Mironescu [3, 4]. Second, we use the following trick: Let ψ be a solution of $\operatorname{div}\psi=\varphi$. Set $U^i := (u_1,\dots,u_{i-1},\psi_i\circ u,u_{i+1},\dots,u_n)$. Then we have

$$\begin{aligned}\det DU^i(x)\mathrm{d}x =&\mathrm{d}u_1(x)\wedge\cdots\wedge\mathrm{d}u_{i-1}(x)\wedge\mathrm{d}(\psi_i\circ u)(x)\wedge\cdots\wedge\mathrm{d}u_n(x)\\ =&\partial_i\psi_i(u(x))\mathrm{d}u_1(x)\wedge\cdots\wedge\mathrm{d}u_n(x)\\ =&\partial_i\psi_i(u(x))\det Du(x)\mathrm{d}x\,.\end{aligned}$$

Hence, we get

$$\begin{aligned}\varphi(u(x))\det Du(x) =&\sum_{i=1}^n\partial_i\psi_i(u(x))\det Du(x)\\ =&\sum_{i=1}^n\det DU^i(x),\end{aligned}\tag{2}$$

which is the sought-for representation as a sum of Jacobian determinants.

We have already noted that by the change of variables formula (1), the integrability of the Brouwer degree is closely related to the weakest space for which we can define the distributional Jacobian determinant $[Ju]$. The question for the weakest space in which $[Ju]$ can be defined has a long history, starting with the work of Morrey[23], Reshetnyak [25] and Ball [1], and with important contributions by many researchers, see e.g. [20, 24, 22, 19, 6, 7, 15], and references therein. In the recent article [5], this question has been answered by the use of fractional Sobolev spaces. In this reference, $[Ju]$ has been defined as an element of the dual of C^1 for $u\in W^{(n-1)/n,n}$. This result contains most of the previously known ones, such as the definition of $[Ju]$ for $u\in W^{1,n-1}\cap L^\infty$ or $u\in W^{1,n^2/(n+1)}$, see [1].

Paralleling the methods from [5], or using the results from [29], one can define $[Ju]$ as an element of $(C^{0,\alpha})^*$ for $u\in C^{0,\alpha}$ and $\alpha>n/(n+1)$. Using this definition, formula (1) has a well defined meaning for $\varphi\in C^1$, since then $\varphi\circ u\in C^{0,\alpha}$. Note however that our treatment using the relation (2), which exploits the special structure of the test function, gives meaning to (1) for a much larger class of test functions. In particular, if we assume that U has Lipschitz boundary, then we will be able to give a well-defined meaning to the left hand side in (1) for $u\in C^{0,\alpha}$ and $\varphi\in L^{p'}$ with $\alpha/p>(n-1)/n$ (where $p^{-1}+(p')^{-1}=1$), which coincides with the right hand side.

The question whether there exist α-Hölder functions whose mapping degree is not in L^p for $n\alpha<pd$ is not addressed here. Note however that for $n\alpha<d$, the image of the boundary $u(\partial U)$ has in general non-vanishing Lebesgue measure, and hence $\deg(u,U,\cdot)$ is

not defined on a set of positive measure (cf. Lemma 2.7).

As a supplement to Theorem 1.1, we show that convergence in $C^{0,\alpha}$ implies convergence of the associated mapping degrees in L^p if $n\alpha > pd$, while for the opposite regime $n\alpha < pd$, there exist sequences that converge to 0 in $C^{0,\alpha}$ whose mapping degrees diverge in L^p.

Theorem 1.2. *Let $0 < \alpha < 1$, $n-1 < d < n$, $1 \le p < \infty$.*

(i) *If $p < \frac{n\alpha}{d}$, $U \subset \mathbb{R}^n$ is open and bounded with $\dim_{\mathrm{box}} \partial U = d$, and $u_k \in C^{0,\alpha}(U;\mathbb{R}^n)$ with $u_k \to u$ in $C^{0,\alpha}(U;\mathbb{R}^n)$, then $\deg(u_k, U, \cdot) \to \deg(u, U, \cdot)$ in $L^p(\mathbb{R}^n)$.*

(ii) *If $p > \frac{n\alpha}{d}$, there exist an open bounded set $U \subset \mathbb{R}^n$ with $\dim_{\mathrm{box}} \partial U = d$, and a sequence $u_k \in C^{0,\alpha}(U;\mathbb{R}^n)$ with $\deg(u_k, U, \cdot) \in L^p(\mathbb{R}^n)$, $u_k \to 0$ in $C^{0,\alpha}(U;\mathbb{R}^n)$ and $\|\deg(u_k, U, \cdot)\|_{L^p} \to \infty$.*

We end this introduction by explaining the plan of the paper. In Section 2, we collect some known methods and theorems that we are going to need in our proofs. They concern (real) interpolation theory, self-similar fractals, the Brouwer degree, the Whitney decomposition of an open subset of $\mathbb{R}^n$, and the relation between the Whitney decomposition and the box dimension. In Section 3, we give the proof of our main result, Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2, using several Lemmas whose proof is given in Section 5.

Notation. The symbol for the non-negative integers is $\mathbb{N} = \{0, 1, \dots\}$. The open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$ will be denoted by $B(x, r)$, while the open ball in $\mathbb{R}^{n-1}$ with center $x \in \mathbb{R}^{n-1}$ and radius r will be denoted by $B^{n-1}(x, r)$. The standard $n-1$ sphere is $\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. The canonical orthonormal basis of $\mathbb{R}^n$ is denoted by $(e_1, \dots, e_n)$. The characteristic function of a set $A \subset \mathbb{R}^n$ is denoted by χ_A.
The n-dimensional Lebesgue measure is denoted by $\mathcal{L}^n$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. The volume of the unit ball in m dimensions is denoted by $\omega_m = \pi^{m/2}/\Gamma(m/2+1)$.
Whenever we want to say that two functions f, g that are defined $\mathcal{L}^n$-almost everywhere on $\mathbb{R}^n$, agree $\mathcal{L}^n$-almost everywhere, then we write $f \doteq g$.
For a Lipschitz function defined on a set $A \subset \mathbb{R}^n$, its Lipschitz constant is $\operatorname{Lip} f = \sup_{x,y\in A, x\neq y} |f(x) - f(y)|/|x-y|$. For sets $A \subset \mathbb{R}^n$ and functions $f : A \to \mathbb{R}$, we set

$$[f]_{C^{0,\alpha}(A)} := \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}\,.$$

If the domain is clear, we often will write $[f]_\alpha \equiv [f]_{C^{0,\alpha}(A)}$ for short. The corresponding Hölder norm is defined by

$$\|f\|_{C^{0,\alpha}(A)} = \sup_{x\in A} |f(x)| + [f]_{C^{0,\alpha}(A)}\,.$$

Let $\Lambda^p\mathbb{R}^n$ denote the set of rank p multi-vectors in $\mathbb{R}^n$, i.e., the linear space

$$\Lambda^p\mathbb{R}^n = \left\{ \sum_{i_1,\dots,i_p\in\{1,\dots,n\}} a_{i_1,\dots,i_p} \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} \,:\, a_{i_1,\dots,i_p} \in \mathbb{R} \right\},$$

With this notation, p-forms are functions on U with values in $\Lambda^p\mathbb{R}^n$. We make $C^{k,\alpha}(U;\Lambda^p\mathbb{R}^n)$ a normed space by setting

$$\|a\|_{C^{k,\alpha}(U;\Lambda^p\mathbb{R}^n)} = \sum_{i_1,\dots,i_p} \|a_{i_1,\dots,i_p}\|_{C^{k,\alpha}(U)}$$

for $a = \sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}$.
The symbol C will have the following meaning: A statement such as $f \leq C(a,b,\dots)g$ means that there exists a numerical constant C that only depends on $a, b, \dots$, such that $f \leq Cg$. The value of C may change from one line to the next.

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2. Preliminaries

2.1. **Tools from interpolation theory.** We are going to use some standard constructions from real interpolation theory, due to Lions and Peetre [17, 18] (see also the textbook [2]). In the following, we give a very short definition of interpolation spaces via the trace method [16].
Let $(E_0, \|\cdot\|_0)$, $(E_1, \|\cdot\|_1)$ be normed spaces. We may equip $E_0 \cap E_1$ and $E_0 + E_1$ with the following norms:

$$\begin{aligned} \|x\|_{E_0 \cap E_1} &= \max\{\|x\|_0, \|x\|_1\} \\ \|x\|_{E_0 + E_1} &= \inf\{\|x_0\|_0 + \|x_1\|_1 : x_0 \in E_0,\, x_1 \in E_1,\, x_0 + x_1 = x\} \end{aligned}$$

Definition 2.1. *For $\theta \in (0,1)$ and $1 \leq p \leq \infty$ we denote by $V(p,\theta,E_1,E_0)$ the set of all functions $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_+, E_0 + E_1)$ with the following properties: $u(t) \in E_1$ and $u'(t) \in E_0$ for all $t > 0$, and with $u_{*,\theta}(t) := t^\theta u(t)$ and $u'_{*,\theta}(t) := t^\theta u'(t)$, we have*

$$u_{*,\theta} \in L^p(\mathbb{R}^+, \mathrm{d}t/t; E_1), \quad u'_{*,\theta} \in L^p(\mathbb{R}^+, \mathrm{d}t/t; E_0)\,.$$

We define a norm on $V = V(p,\theta,E_1,E_0)$ by

$$\|u\|_V := \|u_{*,\theta}\|_{L^p(\mathbb{R}^+,\mathrm{d}t/t;E_1)} + \|u'_{*,\theta}\|_{L^p(\mathbb{R}^+,\mathrm{d}t/t;E_0)}\,.$$

It can be shown that those functions are continuous in $t = 0$ and we define the real interpolation spaces as follows:

Definition 2.2. *The real interpolation space $(E_0, E_1)_{\theta,p}$ is defined as set of traces of functions belonging to $V(p, 1-\theta, E_1, E_0)$ at $t = 0$ together with the norm:*

$$\|x\|^{\mathrm{Tr}}_{(\theta,p)} = \inf\{\|u\|_V : u \in V(p, 1-\theta, E_1, E_0),\, \lim_{t\to 0} u(t) = x\}$$

It can be shown that the Hölder spaces $C^{0,\alpha}(U)$ are identical to the real interpolation space $(C^0(U), C^1(U))_{\alpha,\infty}$, up to equivalence of norms.

2.2. **Self-similar fractals.** We recall the construction of self-similar fractals introduced in [14] (see also [10]). A *similarity* is a map $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $|S(x)-S(y)| = c|x-y|$ for all $x, y \in \mathbb{R}^n$, for some $c > 0$. The number c is called the *ratio* of S. For $i = 1, \dots, k$, let S_i be such a similarity, with ratios smaller than 1. A compact set $K \subset \mathbb{R}^n$ is said to be invariant under $\mathcal{S} = \{S_1, \dots, S_k\}$ if

$$K = \cup_{i=1}^k S_i(K)\,.$$

In fact, one can show that there exists a unique compact set, the attractor set of $\mathcal{S}$, denoted by $K(\mathcal{S})$, that fulfills this property. It consists of the closure of the fixed points of finite compositions of the similarities. A set constructed in this way is called *self-similar.*
For a given set of similarities $\mathcal{S} = \{S_1, \dots, S_k\}$, we define a transformation S on the class of non-empty compact sets by

$$S(E) = \cup_{i=1}^k S_i(E) \tag{3}$$

and write S^l for the l-th iterate of S. For $i_1, \dots, i_l \in \{1, \dots, k\}$ and $E \subset \mathbb{R}^n$, we will use the notation

$$S_{i_1,\dots,i_l}(E) = S_{i_1} \circ \cdots \circ S_{i_l}(E)\,.$$

With this notation, we have

$$S^l(E) = \bigcup_{i_1,\dots,i_l=1}^{k} S_{i_1,\dots,i_l}(E)\,.$$

A convenient way of defining certain self-similar sets in $\mathbb{R}^2$ (i.e., self-similar curves) is by specifying a *generator* for the curve. This is a sequence of points $\gamma : \{1, \dots, k+1\} \to \mathbb{R}^2$ with $|\gamma(1)| < 1$, $|\gamma(i)-\gamma(i-1)| < 1$ for $i = 2, \dots, k+1$. The set of similarities associated to such a generator is given by $\{S_1, \dots, S_k\}$, where S_i is the orientation preserving similarity that maps $(0,0)$ to $\gamma(i)$ and $(1,0)$ to $\gamma(i+1)$. A typical example of a self-similar set constructed from a generator is the Koch curve, see Figure 1. A set of similarities $\mathcal{S}$ is

Figure 1. The Koch curve (right) and its generator (left).

said to satisfy the open set condition if there exists a non-empty open set $V \subset \mathbb{R}^n$ such that

$$\begin{aligned} S_i(V) &\subset V \quad && \text{for } i = 1, \dots, k \\ S_i(V) \cap S_j(V) &= \emptyset \quad && \text{for } i, j = 1, \dots, k,\ i \neq j \end{aligned}$$

The following lemma has been proved in [14, 9]:

Lemma 2.3 (Theorem 9.3 in [9]). *Let $\mathcal{S} = \{S_1, \dots, S_k\}$ be a set of similarities satisfying the open set condition, and let r_i be the ratio of S_i for $i = 1, \dots, k$. Further, let d be the (unique) real number that satisfies*

$$\sum_{i=1}^k r_i^d = 1\,.$$

Then the Hausdorff dimension and box dimension of $K(\mathcal{S})$ agree and are equal to d.

2.3. **Properties of the Brouwer degree.** We recall the definition and some basic properties of the Brouwer degree. For a more thorough exposition with proofs of the claims made here, see e.g. [12].

Let U be a bounded subset of $\mathbb{R}^n$. Further, let $u \in C^\infty(\overline{U};\mathbb{R}^n)$. Assume that $y \in \mathbb{R}^n \setminus u(\partial U)$, and let μ be a C^∞ n-form on $\mathbb{R}^n$ with support in the same connected component of $\mathbb{R}^n \setminus u(\partial U)$ as y, such that $\int_{\mathbb{R}^n} \mu = 1$. Then the degree is defined by

$$\deg(u, U, y) = \int_U u^*(\mu)\,, \tag{4}$$

where u^* is the pull-back by u. It can be shown that this definition is independent of the choice of μ. Further, $\deg(u, U, \cdot)$ is constant on connected components of $\mathbb{R}^n \setminus u(\partial U)$ and integer valued. Moreover, it is invariant under homotopies, i.e., given $H \in C^\infty([0,1] \times \overline{U};\mathbb{R}^n)$ such that $y \notin H([0,1], \partial U)$, we have

$$\deg(H(0,\cdot), U, y) = \deg(H(1,\cdot), U, y)\,.$$

Using these facts, one can go on to define the degree for $u \in C^0(\overline{U};\mathbb{R}^n)$ by approximation. If $u : \overline{U} \to \mathbb{R}^n$ is Lipschitz, and $\mathcal{L}^n(\partial U) = 0$, then it follows from (4) and approximation by smooth functions that

$$\int_{\mathbb{R}^n} \deg(u, U, \cdot)\mu = \int_U u^*(\mu) \tag{5}$$

for any n-form μ on $\mathbb{R}^n$ with coefficients in $L^\infty(\mathbb{R})$. If μ is an exact form, i.e.,

$$\mu = \mathrm{d}\omega$$

for some $n-1$ form ω on $\mathbb{R}^n$, then

$$u^*(\mathrm{d}\omega) = \mathrm{d}\left(u^*\omega\right)\,.$$

If U has Lipschitz boundary, this implies, by Stokes' Theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} \deg(u, U, \cdot)\mathrm{d}\omega &= \int_U \mathrm{d}\left(u^*\omega\right) \\ &= \int_{\partial U} u^*\omega\,. \end{aligned} \tag{6}$$

Assume μ is a given n-form. Since we assume U to be bounded, we may always find some $n-1$-form ω such that $\mathrm{d}\omega = \mu$ on $\operatorname{supp}\deg(u, U, \cdot) \subset u(U)$, and hence (6) shows in particular that the degree only depends on $u|_{\partial U}$. We will write $\deg(u, U, y) = \deg^\partial(u, \partial U, y)$. In the proof of Theorem 1.2 (ii), we will use the following lemma:

Lemma 2.4. *Let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $u : \bar{U} \to \mathbb{R}^n$ be Lipschitz. Further, let $V \subset \partial U$ be relatively open in ∂U, and assume there exists $y_0 \in \mathbb{R}^n$ such that $u(x) = y_0$ for all $x \in \tilde{\partial} V$ (where $\tilde{\partial} V$ denotes the relative boundary of V in ∂U). Define $u_i : \partial U \to \mathbb{R}^n$, $i = 1, 2$ by*

$$u_1(x) = \begin{cases} u(x) & \text{if } x \in V \\ y_0 & \text{if } x \in \partial U \setminus V \end{cases}, \qquad u_2(x) = \begin{cases} y_0 & \text{if } x \in V \\ u(x) & \text{if } x \in \partial U \setminus V \end{cases}.$$

Then

$$\deg^\partial(u, \partial U, y) = \deg^\partial(u_2, \partial U \setminus V, y) + \deg^\partial(u_1, V, y) \quad \text{for all } y \in \mathbb{R}^n \setminus u(\partial U)\,.$$

Proof. We will show

$$\int_{\mathbb{R}^n} \deg^\partial(u,U,\cdot)\mu = \int_{\mathbb{R}^n} \deg^\partial(u_1,U,\cdot)\mu + \int_{\mathbb{R}^n} \deg^\partial(u_2,U,\cdot)\mu$$

for every n-form μ on $\mathbb{R}^n$ with coefficients in L^∞. Indeed, as we remarked below (6), there exists an $n-1$-form ω such that $\mu = d\omega$ on $u(\overline{U})$, and hence

$$\begin{aligned}\int_{\mathbb{R}^n} \deg^\partial(u,U,\cdot)\mu &= \int_{\partial U} u^*\omega \\ &= \int_{\partial U} u_1^*\omega + \int_{\partial U} u_2^*\omega \\ &= \int_{\mathbb{R}^n} \deg^\partial(u_1,U,\cdot)\mu + \int_{\mathbb{R}^n} \deg^\partial(u_2,U,\cdot)\mu\,,\end{aligned}$$

proving the claim of the lemma. □

2.4. **Whitney decomposition and box dimension.** One of our main tools in the proof of Theorem 1.1 will be the Whitney decomposition of an open set U.

Lemma 2.5 (see e.g. [26], Chapter 1, Theorem 3)**.** *Let $U \subset \mathbb{R}^n$ be open. Then there exists a countable collection $W = \{Q_i : i \in \mathbb{N}\}$ of cubes Q_i with the following properties:*

- *For every $Q \in W$, there exist $k, m_1, \dots, m_n \in \mathbb{Z}$ such that $Q = (m_1 2^{-k}, (m_1 + 1)2^{-k}) \times \cdots \times (m_n 2^{-k}, (m_n+1)2^{-k})$. For fixed k, the union of cubes for which this holds for some $m_1, \dots, m_n$ is denoted by W_k.*
- *$U \subset \cup_{Q\in W}\overline{Q}$*
- *The cubes in W are mutually disjoint*
- *$\operatorname{dist}(Q,\partial U) \leq \operatorname{diam} Q \leq 4\operatorname{dist}(Q,\partial U)$ for all $Q \in W$*

Next, we recall the definition of box dimension, and some of its elementary properties.

Definition 2.6. *Let $U \subset \mathbb{R}^n$ be bounded. Let $N_r(U)$ be the number of n-dimensional boxes of side length r that is required to cover U. The box dimension $\dim_{\mathrm{box}} U$ is defined as*

$$\dim_{\mathrm{box}}(U) = \lim_{r\to 0} \frac{\log N_r(U)}{-\log r}\,,$$

if this limit exists.

We also define the β-dimensional Hausdorff-type content for sets $A \subset \mathbb{R}^n$,

$$H^\beta(A) = \lim_{r\to 0}\left(\inf\{k r^\beta : \cup_{i=1}^k B(x_i, r) \supset A\}\right)\,.$$

If $\dim_{\mathrm{box}} A$ exists, then

$$\dim_{\mathrm{box}} A = \inf\{\beta : H^\beta(A) < \infty\}\,,$$

see e.g [9], Definition 3.1.

In the following lemma, for sets $A \subset \mathbb{R}^n$, we will use the notation

$$(A)_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leq \varepsilon\}\,.$$

Lemma 2.7. *Let $V \subset \mathbb{R}^n$ be open and bounded, $U \subset\subset V$, $n-1 < \dim_{\mathrm{box}} \partial U = d < n$, $0 < \alpha < 1$ such that $n\alpha > d$, and $u \in C^{0,\alpha}(V;\mathbb{R}^n)$. Then*

$$\mathcal{L}^n\left(u(\partial U)_\varepsilon\right) \to 0 \qquad \text{as } \varepsilon \to 0\,.$$

Proof. Set $\tilde\varepsilon=\varepsilon^{\alpha^{-1}}$. Choose a finite number of $x_i\in\partial U$, $i=1,\dots,k$, such that

$$\partial U\subset\bigcup_{i=1}^k B(x_i,\tilde\varepsilon)$$
$$B(x_i,\tilde\varepsilon/5)\cap B(x_j,\tilde\varepsilon/5)=\emptyset\quad\text{for } i,j\in\{1,\dots,k\}, i\neq j\,.$$

Such a collection $\{x_i\}$ exists by the Vitali Covering Lemma. Choose $d<\bar d<n\alpha$. This choice implies $H^{\bar d}(\partial U)=0$. By choosing ε small enough, we may assume

$$k\tilde\varepsilon^{\bar d}\leq 1\,,$$

Next observe that

$$\begin{aligned}u(\partial U)\subset&\bigcup_{i=1}^k B\left(u(x_i),\|u\|_{C^{0,\alpha}}\tilde\varepsilon^\alpha\right)\\ =&\bigcup_{i=1}^k B\left(u(x_i),\|u\|_{C^{0,\alpha}}\varepsilon\right)\,.\end{aligned}$$

We set $C^*=\|u\|_{C^{0,\alpha}}+1$ and get

$$(u(\partial U))_\varepsilon\subset\bigcup_{i=1}^k B\left(u(x_i),C^*\varepsilon\right)\,.$$

Hence,

$$\begin{aligned}\mathcal L^n\left((u(\partial U))_\varepsilon\right)\leq& k\mathcal L^n(B(0,1))(C^*\varepsilon)^n\\ \leq& C(u,n)(\tilde\varepsilon^{\alpha n-\bar d})k\tilde\varepsilon^{\bar d}\\ \leq& C(u,n)(\tilde\varepsilon^{\alpha n-\bar d})\\ \to& 0 \text{ as } \varepsilon\to 0\,.\end{aligned}$$

This proves the lemma. □

In the proof of Theorem 1.1, we are going to exploit the following relation between the Whitney decomposition and box dimension:

Theorem 2.8 ([21], Theorem 3.12)**.** *Let $K\subset\mathbb R^n$ be compact, with $\dim_{\rm box}K=d<n$. Let W be the Whitney decomposition of $\mathbb R^n\setminus K$. Then $\lim_{k\to\infty}\frac{\log_2\#W_k}{k}=d$.*

3. Proof of Theorem 1.1

Recall that $C^1(U;\Lambda^{n-1}\mathbb R^n)$ denotes the space of continuously differentiable $n-1$ forms on U. For the subspace of closed forms, we introduce the notation

$$C^1_{\rm cl}(U;\Lambda^{n-1}\mathbb R^n):=\{\omega\in C^1(U;\Lambda^{n-1}\mathbb R^n):\mathrm d\omega=0\}\,.$$

Now we define two norms $\|\cdot\|_{X_0^{n-1}}$, $\|\cdot\|_{X_1^{n-1}}$ on the quotient space $C^1(U;\Lambda^{n-1}\mathbb R^n)/C^1_{\rm cl}(U;\Lambda^{n-1}\mathbb R^n)$:

$$\begin{aligned}\|\omega\|_{X_0^{n-1}}:=&\inf\{\|\omega+\alpha\|_{C^0(U;\Lambda^{n-1}\mathbb R^n)}:\alpha\in C^1(U;\Lambda^{n-1}\mathbb R^n),\mathrm d\alpha=0\}\\ \|\omega\|_{X_1^{n-1}}:=&\|\mathrm d\omega\|_{C^0(U;\Lambda^n\mathbb R^n)}\end{aligned}$$

Let X_0^{n-1}, X_1^{n-1} denote the Banach spaces that one obtains by completion with respect to the above norms respectively.
Next we define a multi-linear operator

$$\begin{aligned} M : C^1(U;\mathbb{R}^n) \to & C^0(U;\Lambda^{n-1}\mathbb{R}^n) \\ (u_1,\dots,u_n) \mapsto & \frac{1}{n}\sum_{i=1}^n (-1)^{i+1} u_i \,\mathrm{d}u_1 \wedge \dots \wedge \widehat{\mathrm{d}u_i} \wedge \dots \wedge \mathrm{d}u_n \,, \end{aligned}$$

where $\widehat{\mathrm{d}u_i}$ denotes omission of the factor $\mathrm{d}u_i$. Note that

$$\mathrm{d}M(u_1,\dots,u_n) = \det Du\,\mathrm{d}x_1 \wedge \dots \mathrm{d}x_n \equiv \det Du\,\mathrm{d}x \,.$$

In the following lemma, let X_θ denote the real interpolation space

$$X_\theta = (X_0^{n-1}, X_1^{n-1})_{\theta,\infty} \,.$$

Lemma 3.1. *Let $U \subset \mathbb{R}^n$ be bounded and open. For $i = 1,\dots,n$, let $\alpha_i \in (0,1)$ such that*

$$\theta := \left(\sum_{i=1}^n \alpha_i\right) - (n-1) > 0 \,.$$

Additionally, let $u = (u_1,\dots,u_n) \in C^1(U;\mathbb{R}^n)$. Then

$$\|M(u_1,\dots,u_n)\|_{X_\theta} \le C(n,\alpha_1,\dots,\alpha_n) \prod_{i=1}^n \|u_i\|_{C^{0,\alpha_i}(U)} \,, \tag{7}$$

Moreover, for $\tilde\theta < \theta$, M extends to a multi-linear operator $C^{0,\alpha_1}(U)\times\dots\times C^{0,\alpha_n}(U) \to X_{\tilde\theta}$.

Notation: All constants C in the proof below may depend on $n, \alpha_1,\dots,\alpha_n$ without explicit statement.

Proof. We use the representation of real interpolation spaces as trace spaces, see Definition 2.1. In particular, we have $C^{0,\alpha_i}(U) = (C^0(U), C^1(U))_{\alpha_i,\infty}$, and hence we may choose $v_i \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; C^0(U))$ with $v_i(t) \in C^1(U)$, $v_i'(t) \in C^0(U)$ for all $t > 0$ such that

$$\begin{aligned} \|t^{1-\alpha_i} v_i(t)\|_{L^\infty(\mathbb{R}^+;C^1(U))} \le & C\|u_i\|_{C^{0,\alpha_i}(U)} \,, \\ \|t^{1-\alpha_i} v_i'(t)\|_{L^\infty(\mathbb{R}^+;C^0(U))} \le & C\|u_i\|_{C^{0,\alpha_i}(U)} \,, \end{aligned}$$

and $u_i = \lim_{t\to 0} v_i(t)$. Then we set

$$w(t) = M(v_1(t),\dots,v_n(t)) \,.$$

By the multi-linearity of M, we have

$$\begin{aligned} \|t^{\sum_i (1-\alpha_i)} w(t)\|_{X_1^{n-1}} \le & C \prod_{i=1}^n \left(t^{1-\alpha_i} \|v_i(t)\|_{C^1(U)}\right) \\ \le & C \prod_{i=1}^n \|u_i\|_{C^{0,\alpha_i}(U)} \end{aligned}$$

and

$$\begin{aligned}
\|t^{\sum_i(1-\alpha_i)}w'(t)\|_{X_0^{n-1}} =&\frac1n t^{\sum_i(1-\alpha_i)}\Bigg\|\sum_{i=1}^n(-1)^{i+1}v_i'(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_n(t)\\
&+\sum_{j\neq i}(-1)^{i+1}v_i(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_j'(t)\wedge\cdots\wedge\mathrm{d}v_n(t)\Bigg\|_{X_0^{n-1}}\\
\le& t^{\sum_i(1-\alpha_i)}\sum_{i=1}^n\left\|(-1)^{i+1}v_i'(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_n(t)\right\|_{X_0^{n-1}}
\end{aligned}\tag{8}$$

where we have used that

$$\begin{aligned}
&\|v_i(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_j'(t)\wedge\cdots\wedge\mathrm{d}v_n(t)\|_{X_0^{n-1}}=\\
&\qquad\qquad\|v_j'(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_n(t)\|_{X_0^{n-1}}\,,
\end{aligned}$$

which in turn is a consequence of

$$\begin{aligned}
&\mathrm{d}\Big(v_i(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_j'(t)\wedge\cdots\wedge\mathrm{d}v_n(t)\Big)\\
&\qquad\qquad=\mathrm{d}\left((-1)^{j+i}v_j'(t)\mathrm{d}v_1(t)\wedge\cdots\wedge\widehat{\mathrm{d}v_i(t)}\wedge\cdots\wedge\mathrm{d}v_n(t)\right)\,.
\end{aligned}$$

From (8) we get

$$\begin{aligned}
\|t^{\sum_i(1-\alpha_i)}w'(t)\|_{X_0^{n-1}}\le& C\sum_{i=1}^n\left(t^{1-\alpha_i}\|v_i'(t)\|_{C^0(U)}\right)\prod_{j\neq i}\left(t^{1-\alpha_j}\|v_j(t)\|_{C^1(U)}\right)\\
\le& C\prod_{i=1}^n\|u_i\|_{C^{0,\alpha_i}(U)}\,.
\end{aligned}$$

Hence $w(0)\in(X_0^{n-1},X_1^{n-1})_{1-\sum_i(1-\alpha_i),\infty}=X_\theta$, with $\|w(0)\|_{X_\theta}\le C\prod_i\|u_i\|_{C^{0,\alpha_i}}$. The estimate (7) follows from $w(0)=M(u_1,\dots,u_n)$.
To prove the statement about the extension, we choose $\beta_i<\alpha_i$ with $\sum_i\beta_i=n-1+\tilde\theta$. Then we have

$$\|M(u_1,\dots,u_n)\|_{X_{\tilde\theta}}\le\prod_{i=1}^n\|u_i\|_{C_{0,\beta_i}}\tag{9}$$

for $u=(u_1,\dots,u_n)\in C^1(U;\mathbb{R}^n)$. Now every $\tilde u=(\tilde u_1,\dots,\tilde u_n)\in C^{0,\alpha_1}\times\cdots\times C^{0,\alpha_n}$ can be approximated in $C^{0,\beta_1}\times\cdots\times C^{0,\beta_n}$ by sequences of functions in $C^1(U;\mathbb{R}^n)$, and hence the existence of a unique extension follows from (9). □

3.1. **Integrating distributional Jacobians over sets with fractal boundary.** The purpose of the construction of the interpolation space $X_\theta=(X_0^{n-1},X_1^{n-1})_{\theta,\infty}$ has been to make its elements suitable for integration over fractals of dimension up to (but not including) $n-1+\theta$. The corresponding definition will be given in the present subsection. This will be similar to the constructions in [13].

In the following, let $U \subset \mathbb{R}^n$ be fixed, with $d := \dim_{\mathrm{box}} \partial U < n - 1 + \theta$. Let W be the Whitney decomposition of U, cf. Section 2.4. Since $M \in X_\theta$, there exists $\tilde{M}(\cdot) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; X_0^{n-1})$ with $\tilde{M}(t) \in X_1^{n-1}$, $\tilde{M}'(t) \in X_0^{n-1}$ for all $t > 0$ such that

$$t^{1-\theta} \left(\|\tilde{M}(t)\|_{X_1^{n-1}} + \|\tilde{M}'(t)\|_{X_0^{n-1}} \right) \leq \|M\|_{X_\theta} \quad \text{for all } t \in \mathbb{R}^+ ,$$

and

$$\lim_{t \to 0} \|M - \tilde{M}(t)\|_{X_0^{n-1}} = 0 ,$$

see Section 2.1.

Definition 3.2. *Assume that $n-1+\theta > d$. For $M \in X_\theta$ let the integral $\int_U \mathrm{d}M$ be defined by*

$$\int_U \mathrm{d}M := \sum_{Q \in W} \int_Q \mathrm{d}\tilde{M}(\operatorname{diam} Q) + \int_{\partial Q} (M - \tilde{M}(\operatorname{diam} Q)) ,$$

where $\tilde{M} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+; X_0^{n-1})$ is chosen as above.

Lemma 3.3. *The above definition makes $\int_U \mathrm{d}M$ well defined for $M \in X_\theta$, and the map*

$$M \mapsto \int_U \mathrm{d}M$$

is continuous on X_θ with $|\int_U \mathrm{d}M| \leq C(U)\|M\|_{X_\theta}$.

Proof. Let $\tilde{M}(\cdot)$ as above, and let $Q \in W$. First we estimate

$$\begin{aligned} \left| \int_Q \mathrm{d}\tilde{M}(\operatorname{diam} Q) \right| &\leq \mathcal{L}^n(Q) \|\tilde{M}(\operatorname{diam} Q)\|_{X_1^{n-1}} \\ &\leq \mathcal{L}^n(Q) (\operatorname{diam} Q)^{\theta - 1} \|M\|_{X_\theta} . \end{aligned}$$

To estimate $\int_{\partial Q} (M - \tilde{M}(\operatorname{diam} Q))$, we first note that

$$\begin{aligned} \|M - \tilde{M}(\operatorname{diam} Q)\|_{X_0^{n-1}} &\leq \int_0^{\operatorname{diam} Q} \|\tilde{M}'(t)\|_{X_0^{n-1}} \\ &\leq C \int_0^{\operatorname{diam} Q} t^{\theta - 1} \|M\|_{X_\theta} \\ &\leq C (\operatorname{diam} Q)^\theta \|M\|_{X_\theta} . \end{aligned}$$

Hence we get

$$\begin{aligned} \left| \int_{\partial Q} (M - \tilde{M}(\operatorname{diam} Q)) \right| &\leq \mathcal{H}^{n-1}(Q) \|M - \tilde{M}(\operatorname{diam} Q)\|_{X_0^{n-1}} \\ &\leq C \mathcal{H}^{n-1}(Q) (\operatorname{diam} Q)^\theta \|M\|_{X_\theta} . \end{aligned}$$

By Theorem 2.8 the number of cubes in W of sidelength 2^{-k} can be estimated by $C 2^{kd}$, where the constant C may depend on the domain U, and $d = \dim_{\mathrm{box}} \partial U$. In this way we obtain

$$\begin{aligned} \left| \int_U \mathrm{d}M \right| &\leq \sum_{Q \in W} \mathcal{L}^n(Q) (\operatorname{diam} Q)^{\theta - 1} \|M\|_{X_\theta} + \mathcal{H}^{n-1}(Q) (\operatorname{diam} Q)^\theta \|M\|_{X_\theta} \\ &\leq C \sum_{k \in \mathbb{N}} 2^{dk} 2^{-(n-1)k} 2^{-\theta k} \|M\|_{X_\theta} . \end{aligned}$$

By the assumption $d < n-1+\theta$ the infinite sum converges absolutely. This proves

$$\left|\int_U \mathrm{d}M\right| \leq C\|M\|_{X_\theta}\,,$$

and in particular it follows that $\int_U \mathrm{d}M$ does not depend on the choice of $\tilde{M}$, which makes the integral well defined. Also, the continuity of $M \mapsto \int_U \mathrm{d}M$ as a map from X_θ to $\mathbb{R}$ follows by linearity. □

We are ready to prove Theorem 1.1. In the proof below, all constants C may depend on n, α, p, d without explicit statement.

Proof of Theorem 1.1. In this proof we assume $p > 1$, and define p' by requiring $p^{-1} + (p')^{-1} = 1$. Note that by assumption, we have $p < n/(n-1)$ and hence $p' > n$.
Using the representation of Hölder spaces as trace spaces (cf. Definition 2.1), we can choose $v_i : \mathbb{R}^+ \to C^1(U)$, $i \in \{1, \dots, n\}$, such that

$$\|t^{1-\alpha}v_i(t)\|_{L^\infty(\mathbb{R}^+;C^1(U))} + \|t^{1-\alpha}v_i'(t)\|_{L^\infty(\mathbb{R}^+;C^0(U))} \leq C\|u\|_{C^{0,\alpha}(U)}$$

and $\lim_{t\to 0} v_i(t) = u_i$ in $C^0(U)$.
We write $v(t) = (v_1(t), \dots, v_n(t))$, and claim that

$$\begin{aligned}\limsup_{t\to 0}\sup\left\{\int_{\mathbb{R}^n}\deg(v(t),U,y)\varphi(y)\mathrm{d}y : \varphi \in L^{p'}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n),\ \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \leq 1\right\}\\ <C\left(\|u\|_{C^{0,\alpha}(U;\mathbb{R}^n)}\right)^{n/p}\,,\end{aligned} \tag{10}$$

where the constant on the right hand side may depend on U. From the estimate (10) it follows that $\{\deg(v(t), U, \cdot) : t \leq 1\}$ is bounded and hence precompact in $L^p(\mathbb{R}^n)$. By $v(t) \to u$ in C^0, it follows the pointwise convergence $\deg(v(t), U, \cdot) \to \deg(u, U, \cdot)$ on $\mathbb{R}^n \setminus u(\partial U)$. By Lemma 2.7, we have $\mathcal{L}^n(u(\partial U)) = 0$, and hence $\deg(v(t), U, \cdot) \to \deg(u, U, \cdot)$ almost everywhere. In combination with the compactness in L^p, it follows $\deg(v(t), U, \cdot) \rightharpoonup \deg(u, U, \cdot)$ in $L^p(\mathbb{R}^n)$. In particular, $\deg(u, U, \cdot) \in L^p$ with $\|\deg(u, U, \cdot)\|_{L^p} < C\|u\|^n_{C^{0,\alpha}}$. Since the support of $\deg(u, U, \cdot)$ is bounded, we also have $\deg(u, U, \cdot) \in L^1(\mathbb{R}^n)$, and the theorem is proved. It remains to show (10).
Let $\varphi \in L^{p'}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with $\|\varphi\|_{L^{p'}} \leq 1$. Let $\zeta \in W^{2,p'}(\mathbb{R}^n)$ be the solution of

$$\Delta\zeta = \varphi \quad \text{ on } \mathbb{R}^n\,,$$

and define $\psi \in W^{1,p'}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ by

$$\psi(x) = D\zeta(x) - D\zeta(0)\,.$$

By standard estimates, we have $\|D\psi\|_{L^{p'}} \leq C\|\varphi\|_{L^{p'}}$. Hence by Morrey's inequality, we have

$$[\psi]_{C^{0,1-n/p'}(U)} \leq C\|\varphi\|_{L^{p'}}\,.$$

Now since $\psi(0) = 0$ we have for any $w \in C^{0,\alpha}(U;\mathbb{R}^n)$

$$\sup_{x\in U}|\psi\circ w(x)| \leq [\psi]_{C^{0,1-n/p'}}\left(\sup_{x\in U}|w(x)|\right)^{1-n/p'} \leq [\psi]_{C^{0,1-n/p'}}\|w\|^{1-n/p'}_{C^{0,\alpha}}\,.$$

Furthermore for $x, y \in U$, we have

$$\begin{aligned}|\psi\circ w(x) - \psi\circ w(y)| \leq& [\psi]_{C^{0,1-n/p'}}|w(x)-w(y)|^{1-n/p'}\\ \leq& [\psi]_{C^{0,1-n/p'}}\|w\|^{1-n/p'}_{C^{0,\alpha}}|x-y|^{(1-n/p')\alpha}\,.\end{aligned}$$

Let $\tilde\alpha := \alpha(1-n/p')$. By the above, we have for all $t>0$,

$$\begin{aligned}\psi\circ v(t) \in& C^{0,\tilde\alpha}(U;\mathbb{R}^n)\,,\\ \|\psi\circ v(t)\|_{C^{0,\tilde\alpha}(U)} \leq& C\|\varphi\|_{L^{p'}}\|v(t)\|^{1-n/p'}_{C^{0,\alpha}(U)}\\ \leq& C\|\varphi\|_{L^{p'}}\|u\|^{1-n/p'}_{C^{0,\alpha}(U)}\,,\end{aligned}\tag{11}$$

where in the last estimate, we have assumed that t is small enough and used Lemma A.2. Next, for $i=1,\dots,n$, we set $\tilde v_i(t)=\psi_i\circ v(t)$, and

$$V^j(t) := (v_1(t),\dots,v_{j-1}(t),\tilde v_j(t),v_{j+1}(t),\dots,v_n(t))\ .$$

For $t>0$, we have

$$\left|\int_{\mathbb{R}^n}\varphi(y)\deg(v(t),U,y)\mathrm{d}y\right| = \left|\int_U \varphi(v(t)(x))\det Dv(t)(x)\mathrm{d}x\right|\ . \tag{12}$$

Using the relation $\mathrm{d}M(V^j(t)) = \det DV^j(t)\mathrm{d}x$, we get

$$\begin{aligned}\sum_{j=1}^n \mathrm{d}M(V^j(t))|_x =& \sum_{j=1}^n \det DV^j(t)|_x\mathrm{d}x\\ =& \sum_{j=1}^n \mathrm{d}V_1^j(t)|_x\wedge\cdots\wedge\mathrm{d}V_n^j(t)|_x\\ =& \sum_{j=1}^n \partial_j\psi_j|_{v(t)(x)}\,\mathrm{d}v_1(t)|_x\wedge\cdots\wedge\mathrm{d}v_n(t)|_x\\ =& \varphi(v(t)(x))\det Dv(t)(x)\mathrm{d}x\,.\end{aligned}$$

Inserting this into (12), we get

$$\left|\int_{\mathbb{R}^n}\varphi(y)\deg(v(t),U,y)\mathrm{d}y\right| = \left|\int_U\sum_{j=1}^n\mathrm{d}M(V^j(t))\right|\ .$$

We set

$$\begin{aligned}\bar\theta :=& \theta-d+n-1\\ =& (1-n/p')\alpha+(n-1)\alpha-d\\ =& \frac{n\alpha}{p}-d\,.\end{aligned}$$

Note that by $p<\frac{n\alpha}{d}$, we have $\bar\theta>0$. Now we apply Lemmas 3.3 and 3.1 to obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi(y)\deg(v(t),U,y)\mathrm{d}y\right| \leq& \sum_j\left|\int_U\mathrm{d}M(V^j(t))\right|\\ \leq& C\sum_j\|M(V^j(t))\|_{X_{\bar\theta}}\\ \leq& C\|v(t)\|^{n/p}_{C^{0,\alpha}}\\ \leq& C\|u\|^{n/p}_{C^{0,\alpha}}\end{aligned}$$

where we have also used Lemma A.2 in the last estimate. This proves (10) and hence the theorem. □

Remark 3.4. *By the method of proof we are using, we cannot get the estimate* $\|deg(u, U, \cdot)\|_{L^1} \leq C\|u\|^n_{C^{0,\alpha}}$*, since this would require* $W^{2,\infty}$ *estimates on the solution of* $\Delta\zeta = \varphi$ *with* $\|\varphi\|_{L^\infty} \leq 1$*, which of course do not hold in general.*

4. Proof of Theorem 1.2

4.1. **Proof of Theorem 1.2 (i).** The first part of the theorem is just a corollary to Theorem 1.1.

Proof of Theorem 1.2 (i). Let $p < q < \frac{n\alpha}{d}$. By Theorem 1.1, we have

$$\| \deg(u_k, U, \cdot)\|_{L^q} \leq C(n, \alpha, q, d)\|u\|^n_{C^{0,\alpha}} ,$$

and hence $\deg(u_k, U, \cdot)$ is weakly compact in L^q. In particular, $|\deg(u_k, U, \cdot)|^p$ is equi-integrable. To show the strong convergence $\deg(u_k, U, \cdot) \to \deg(u, U, \cdot)$ in L^p, it is sufficient to show $\deg(u_k, U, \cdot) \to \deg(u, U, \cdot)$ in measure, i.e., for every $\delta > 0$,

$$\mathcal{L}^n\left(\{y : |\deg(u_k, U, y) - \deg(u, U, y)| > \delta\}\right) \to 0 \text{ as } k \to \infty .$$

Since the Brouwer degree is integer-valued, this is equivalent to

$$\mathcal{L}^n\left(\{y : \deg(u_k, U, y) \neq \deg(u, U, y)\}\right) \to 0 \text{ as } k \to \infty . \tag{13}$$

Indeed, let $\varepsilon > 0$, and choose k_0 large enough that $\sup |u - u_k| < \varepsilon/2$ for $k > k_0$. Then

$$y \notin \{tu(x) + (1 - t)u_k(x) : t \in [0, 1], x \in \partial U\} \qquad \text{for all } y \in \mathbb{R}^n \setminus (u(\partial U))_\varepsilon , k > k_0 .$$

By the homotopy invariance of the degree, this implies

$$\deg(u_k, U, y) = \deg(u, U, y) \qquad \text{for all } y \in \mathbb{R}^n \setminus (u(\partial U))_\varepsilon , k > k_0 .$$

The claim (13) now follows from Lemma 2.7. This proves (i). □

4.2. **Proof of Theorem 1.2 (ii).** The present section and Section 5 are devoted to the proof of Theorem 1.2 (ii). It consists of a rather explicit construction of an example.
The basic idea is that one considers sequences u_m of functions defined on a self-similar set of given box dimension d (which is the boundary of some open set U). As the index m increases, the functions u_m use smaller and smaller scales of the self-similar set ∂U to develop "loops". Each of these loops increases the degree, and has (locally) controlled $\tilde{\alpha}$-Hölder semi-norm, where $\tilde{\alpha}$ is slightly larger than α. Thus one constructs a sequence that converges to 0 in $C^{0,\alpha}$ for which the L^p norm of the degree diverges. For the reader's convenience, we first outline the strategy of proof in a little more detail.

1. In Lemma 4.1 and 4.2, we construct the self-similar set ∂U and the pre-fractals ∂U^m, that will be helpful for the definition of the loops at scale m. To lift maps defined on ∂U^m to ∂U, we define certain projection maps (see Lemma 4.4).
2. Then we define "single loops" (Lemma 4.5). These are defined on $(n-1)$-dimensional boxes of sidelength one. Also, we find collections of disjoint $(n - 1)$-dimensional boxes on ∂U^m of sidelength r^m (Lemma 4.8). We work with Euclidean motions and rescalings to lift the "single loop" to each of these boxes, such that the resulting map will have controlled $\tilde{\alpha}$-Hölder semi-norm (see Definition 4.6 and Notation 4.9), where $\tilde{\alpha}$ is slightly larger that α.

3. We then use the compact embedding between Hölder spaces to show that these functions converge to 0 in $C^{0,\alpha}$, while we may use Lemma 2.4 to show that the associated Brouwer degrees diverge in L^p.

From now on, we assume $\tilde{\alpha}$ to be fixed such that

$$\alpha < \tilde{\alpha} < \frac{np}{d}\,. \tag{14}$$

We collect some useful notation. Firstly, we set

$$\begin{aligned} L =& \{(x,0) \in \mathbb{R}^2 : x \in [0,1]\}\,, \\ D =& \{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\, 0 < y < \min(x, 1-x)\}\,. \end{aligned}$$

Lemma 4.1. *Let $1 < \bar{d} < 2$. Then there exist $r > 0$, $N \in \mathbb{N}$, and for each $i \in \{1, \dots, N\}$ a similarity $S_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that the following properties are fulfilled:*

(i) *$\mathcal{H}^1(S_i(L)) = r$ for $i = 1, \dots, N$.*
(ii) *The union $\cup_{i=1}^N S_i(L)$ is the image of a continuous curve with start point $(0,0)$ and end point $(1,0)$.*
(iii) *$Nr^{\bar{d}} = 1$.*
(iv) *For $i, j \in \{1, \dots, N\}$, we have*

$$\begin{aligned} S_i(D) &\subset D\,, \\ S_i(D) \cap S_j(D) &= \emptyset \qquad \text{if } i \neq j\,, \\ S_i(\overline{D}) \cap S_j(\overline{D}) &= \emptyset \qquad \text{if } |i-j| > 1\,. \end{aligned}$$

(v) *$r < \frac{1}{2}$ and $2r^{1-\alpha} \leq 1$.*

The proof can be found in Section 5.

For the rest of this section, let d, n be fixed with $n-1 \leq d < n$. Further, let $\bar{d} = d-(n-2)$ and fix N, r and a set of similarities $\mathcal{S} = \{S_1, \dots, S_N\}$ as in Lemma 4.1. In the following, we are going to use the notation introduced in Section 2.2.

We now define four (orientation preserving) Euclidean motions $S_1^*, \dots, S_4^* : \mathbb{R}^2 \to \mathbb{R}^2$ by their actions on $(0,0)$ and $(1,0)$,

$$\begin{array}{llll} S_1^*(0,0) = & (0,1) & S_1^*(1,0) = & (1,1) \\ S_2^*(0,0) = & (1,1) & S_2^*(1,0) = & (1,0) \\ S_3^*(0,0) = & (1,0) & S_3^*(1,0) = & (0,0) \\ S_4^*(0,0) = & (0,0) & S_4^*(1,0) = & (0,1)\,. \end{array}$$

Next, for $i_0 \in \{1, \dots, 4\}$ and $i_1, \dots, i_m \in \{1, \dots, N\}$, we introduce the notation

$$S_{i_0|i_1,\dots,i_m} = S_{i_0}^* \circ S_{i_1} \circ \dots \circ S_{i_m}\,.$$

Definition 4.2. *For $m \in \mathbb{N}$, let $\tilde{U}^m \subset \mathbb{R}^2$ be the bounded open set with boundary*

$$\begin{aligned} \partial\tilde{U}^m :=& \bigcup_{i_0=1}^{4} S_{i_0}^* S^m(L) \\ =& \bigcup_{\substack{i_0 \in \{1,\dots,4\} \\ i_1,\dots,i_m \in \{1,\dots,N\}}} S_{i_0|i_1,\dots,i_m}(L)\,. \end{aligned} \tag{15}$$

Further, let $\tilde{U} \subset \mathbb{R}^2$ be the bounded open set with boundary

$$\partial\tilde{U} = \bigcup_{i_0=1}^{4} S^*_{i_0} K(\mathcal{S}), \tag{16}$$

where $K(\mathcal{S})$ is the attractor set of $\mathcal{S}$, cf. Section 2.2. For $m \in \mathbb{N}$, we define $U^m = \tilde{U}^m \times (0,1)^{n-2}$ and moreover, $U = \tilde{U} \times (0,1)^{n-2}$.

For a sketch of $\partial U^0, \partial U^1$ and ∂U^4 (with $n = 2$, $N = 5$) see Figure 2.

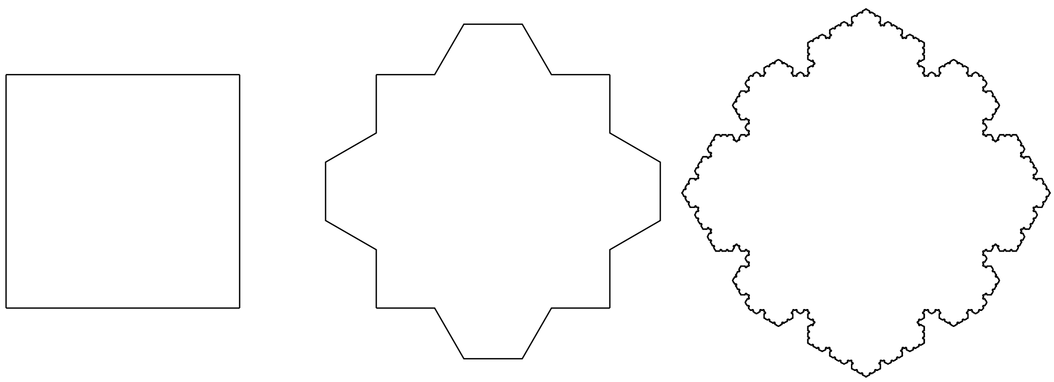

FIGURE 2. The pre-fractals $\partial U^0, \partial U^1$ and ∂U^4 (with $n = 2$, $N = 5$).

Remark 4.3. *The sets $\tilde{U}^m$ and $\tilde{U}$ are well defined, since the right hand sides in* (15), (16) *are closed curves by Lemma 4.1 (ii) and the definition of the Euclidean motions S^*_i, $i = 1, \dots, 4$. Also note that by Lemma 4.1 (iii) and (iv) and Lemma 2.3 we have* $\dim_{\text{box}} \partial U = d$.

We will need certain "projection maps" to pull back maps defined on the pre-fractals ∂U^m to the fractal ∂U:

Lemma 4.4. *For every $m \in \mathbb{N}$, there exist Lipschitz maps $P^m_{m+1} : \overline{U^{m+1}} \setminus U^m \to \partial U^m$ and $P^m : \overline{U} \to \overline{U^m}$ with the following properties:*

- $\mathrm{Lip}(P^m_{m+1}) \leq 1$, $\mathrm{Lip}(P^m) \leq 1$.
- *If $z = (z', z'') \in \overline{U^{m+1}}$ with $z' \in \mathbb{R}^2$, $z'' \in \mathbb{R}^{n-2}$, then $P^m_{m+1}(z) = (\bar{z}, z'')$ for some $\bar{z} \in \mathbb{R}^2$.*

Again, the proof is postponed to Section 5.
In the statement of the next lemma, we set $B := B^{(n-1)}(0,1) \times \{0\}$, and by slight abuse of notation, we write $\partial B := \big(\partial B^{(n-1)}(0,1)\big) \times \{0\}$.

Lemma 4.5. *There exists a Lipschitz map $\tilde{\zeta} : \overline{B} \to \mathbb{R}^n$ (whose Lipschitz constant only depends on n) with the following properties:*

(i) *$\tilde{\zeta}(B) \subset \mathbf{S}^{n-1}$ and $\tilde{\zeta}(x) = -e_n$ for all $x \in \partial B$.*
(ii) *If $W \subset \mathbb{R}^n$ is open and bounded with Lipschitz boundary such that*

$$\overline{B} \subset \partial W$$

and the outer normal to W on B is e_n, then $\zeta^{(W)} : \partial W \to \mathbb{R}^n$ defined by

$$\zeta^{(W)}(x) = \begin{cases} \tilde{\zeta}(x) & \text{if } x \in \overline{B} \\ -e_n & \text{else} \end{cases}$$

satisfies

$$\deg^\partial(\zeta^{(W)}, \partial W, y) \doteq \begin{cases} 1 & \text{if } y \in B(0,1) \\ 0 & \text{else.} \end{cases}$$

Again, the proof is postponed to Section 5.

Definition 4.6. *Let $\rho > 0$ and let $W \subset \mathbb{R}^n$ be an open bounded Lipschitz set with $B_\rho := B^{n-1}(0, \rho) \times \{0\} \subset \partial W$, such that the outer normal of W on B_ρ is e_n. Then we define a Lipschitz map $\zeta_\rho^{(W)} : \partial W \to \mathbb{R}^n$ by*

$$\zeta_\rho^{(W)}(x) = \begin{cases} \rho^{\tilde{\alpha}} \tilde{\zeta}(x/\rho) & \text{if } x \in \overline{B_\rho} \\ -\rho^{\tilde{\alpha}} e_n & \text{else,} \end{cases}$$

where $\tilde{\zeta}$ has been defined in Lemma 4.5.

See Figure 3 for a sketch of $\zeta_\rho^{(W)}$. From now on, we are going to drop the superscript (W) for ease of notation, and write $\zeta_\rho^{(W)} \equiv \zeta_\rho$.

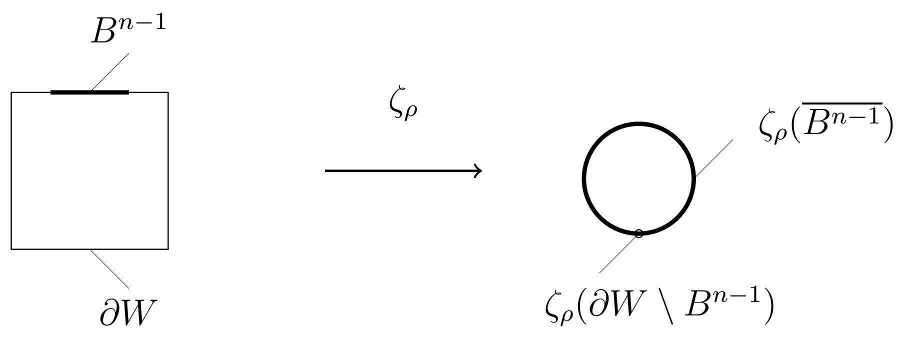

Figure 3. A sketch of the construction from Definition 4.6. In the figure, $B^{n-1} \equiv B^{n-1}(0, \rho/2) \times \{0\}$.

Remark 4.7. (i) *As a consequence of Lemma 4.5 (i) and (ii), ζ_ρ is indeed a well defined Lipschitz map with*

$$\operatorname{Lip} \zeta_\rho \le C(n) \rho^{\tilde{\alpha}-1} .$$

(ii) *By Lemma 4.5 (ii), we have*

$$\deg^\partial(\zeta_\rho, \partial W, y) \doteq \begin{cases} 1 & \text{if } y \in B(0, \rho^{\tilde{\alpha}}) \\ 0 & \text{else.} \end{cases}$$

In the next lemma, by an "$(n-1)$-dimensional box", we mean the image of $[-\rho, \rho]^{n-1} \times \{0\}$ under some Euclidean motion, for some $\rho > 0$.

Lemma 4.8. *For every $m \in \mathbb{N}$, there exists a finite collection $\mathcal{Q}^m$ of $(n-1)$-dimensional boxes of sidelength r^m, such that, writing*

$$\mathcal{Q}^m = \bigcup_{i=1}^{\#\mathcal{Q}^m} Q_i^m\,,$$

the following holds:

(i) *For every $i \in \{1, \dots, \#\mathcal{Q}^m\}$, we have $Q_i^m \subset \partial\tilde{U}^m \times [0,1]^{n-2}$, with $\tilde{U}^m$ as in Definition 4.2.*

(ii) *If $i, j \in \{1, \dots, \#\mathcal{Q}^m\}$, $i \neq j$, then $Q_i^m \cap Q_j^m = \emptyset$. If additionally $\overline{Q_i^m} \cap \overline{Q_j^m} = \emptyset$, then $\operatorname{dist}(Q_i^m, Q_j^m) \geq r^m$.*

(iii) $\lim_{m\to\infty} N^{-m} r^{(n-2)m}\, \#\mathcal{Q}^m = 1\,.$

Again, the proof is postponed to Section 5.

Notation 4.9. *For $m \in \mathbb{N}$ and $x \in \partial U^m$, let $\nu^m(x)$ denote the outward normal to ∂U^m at x, if it exists. Let $\mathcal{N}^m \subset \partial U^m$ denote the set of points for which $\nu^m(x)$ does not exist. For $i \in \{1, \dots, \#\mathcal{Q}^m\}$, let E_i^m denote a Euclidean motion that satisfies*

$$\begin{aligned} E_i^m((-r^m/2, r^m/2)^{n-1} \times \{0\}) =& Q_i^m \\ E_i^m(e_n) =& E_i^m(0) + \nu^m(E_i^m(0))\,, \end{aligned}$$

and let B_i^m be the $(n-1)$-ball of radius $r^m/2$ at the center of Q_i^m, i.e.,

$$B_i^m := E_i^m(B^{n-1}(0, r^m/2) \times \{0\})\,.$$

We are now ready to prove the second part of Theorem 1.2.

Proof of Theorem 1.2 (ii). Let $U, U^m \subset \mathbb{R}^n$ be as in Definition 4.2, for $m \in \mathbb{N}$. For $x \in \partial U^m$, we set

$$v_m(x) = \begin{cases} \zeta_{r^m/2}((E_i^m)^{-1}(x)) & \text{if } x \in B_i^m \\ -(r^m/2)^{\tilde{\alpha}} e_n & \text{if } x \in \partial U^m \setminus \left(\bigcup_{i=1}^{\#\mathcal{Q}^m} B_i^m\right)\,, \end{cases}$$

where we used the notation introduced in Definition 4.6 and Notation 4.9. We immediately see that v_m is Lipschitz with

$$\sup_{x\in\partial U} |v_m| \leq C r^{\tilde{\alpha} m}\,. \tag{17}$$

Now let $\alpha < \alpha' < \tilde{\alpha}$. By (17), we have

$$\sup_{x\in\partial U} |v_m| \leq \varepsilon_m r^{\alpha' m} \tag{18}$$

with $\varepsilon_m := C r^{m(\tilde{\alpha}-\alpha')}$. Next, for $|x-y| > r^m$, we have

$$|v_m(x) - v_m(y)| \leq 2\varepsilon_m r^{\alpha' m} \leq 2\varepsilon_m |x-y|^{\alpha'}\,. \tag{19}$$

From Remark 4.7 (i), we have $\operatorname{Lip}(v_m) \leq C r^{m(\tilde{\alpha}-1)}$. Hence, for $|x-y| \leq r^m$, we have

$$|v_m(x) - v_m(y)| \leq C|x-y|\varepsilon_m r^{m(\alpha'-1)} \leq C\varepsilon_m |x-y|^{\alpha'}\,. \tag{20}$$

By (18), (19) and (20), we have

$$\|v_m\|_{C^{0,\alpha'}(\partial U^m)} < C\varepsilon_m \to 0 \quad \text{as } m \to \infty\,. \tag{21}$$

We come to the computation of $\deg^\partial(v_m, \partial U^m, \cdot)$. To do so, we introduce some additional notation. For $m \in \mathbb{N}$, We set

$$B^{*,m} = B(0, (r^m/2)^{\tilde{\alpha}})$$

and for $i = 1, \ldots, \#\mathcal{Q}^m$, we define $\zeta_i^m : \partial U^m \to \mathbb{R}^n$ by

$$\zeta_i^m(x) = \begin{cases} \zeta_{r^m/2}((E_i^m)^{-1}(x)) & \text{if } x \in B_i^m \\ -(r^m/2)^{\tilde{\alpha}} e_n & \text{else.} \end{cases}$$

We note that by Remark 4.7 (ii), we have

$$\begin{aligned} \deg^\partial(\zeta_i^m, \partial U^m, x) \doteq & \begin{cases} 1 & \text{if } x \in B^{*,m} \\ 0 & \text{else} \end{cases} \\ =& \chi_{B^{*,m}} . \end{aligned}$$

By repeated application of Lemma 2.4, with $V := B_i^m$ for $i \in \{1, \ldots, \#\mathcal{Q}^m\}$,

$$\begin{aligned} \deg(v_m, U^m, \cdot) \doteq & \sum_{i=1}^{\#\mathcal{Q}^m} \deg^\partial(\zeta_i^m, \partial U^m, \cdot) \\ \doteq & \#\mathcal{Q}^m \, \chi_{B^*} , \end{aligned}$$

Hence,

$$\| \deg(v_m, U^m, \cdot)\|_{L^p}^p = (\#\mathcal{Q}^m)^p \, \omega_n (r^m/2)^{n\tilde{\alpha}} .$$

Using Lemma 4.8 (iii) and the relation between r and N from Lemma 4.1 (iii), we have

$$\begin{aligned} \lim_{m\to\infty} \| \deg(v_m, U^m, \cdot)\|_{L^p} =& \lim_{m\to\infty} N^m r^{-(n-2)m} \omega_n^{1/p} (r^m/2)^{n\tilde{\alpha}/p} \\ \geq & \lim_{m\to\infty} C(n, p, \tilde{\alpha}) r^{m(-d+\tilde{\alpha}n/p)} \\ =& +\infty . \end{aligned}$$

We define $u_m : \partial U \to \mathbb{R}^n$ by

$$u_m(x) = v_m(P^m(x)) ,$$

and extend u_m from ∂U to $\overline{U}$ by Theorem A.1, such that

$$\|u_m\|_{C^{0,\alpha'}(U)} < C \|v_m \circ P_m\|_{C^{0,\alpha'}(\partial U^m)} \leq C\varepsilon_m \to 0 .$$

By the compact embedding $C^{0,\alpha'}(U) \to C^{0,\alpha}(U)$,

$$u_m \to 0 \quad \text{in } C^{0,\alpha}(U) .$$

Furthermore, note that

$$\deg^\partial(u_m, \partial U, \cdot) \doteq \deg^\partial(v_m, \partial U^m, \cdot) \qquad \text{for all } m \in \mathbb{N}$$

and hence $\|\deg^\partial(u_m, \partial U, \cdot)\|_{L^p} \to \infty$. This proves the theorem. $\square$

5. Proof of Lemmas used in Section 4

Proof of Lemma 4.1. We first construct an auxiliary sequence of points in $\mathbb{R}^2$, depending on two parameters $\beta \in (0, \pi/2]$ and $M \in \mathbb{N} \setminus \{0\}$. From this sequence of points, we will construct a generator for a self-similar fractal later in the proof.
First, let

$$\begin{aligned} e^\beta_{\text{up}} :=& (\sin\beta, \cos\beta) \\ e^\beta_{\text{down}} :=& (\sin\beta, -\cos\beta) \\ e^\beta_{\text{right}} :=& (1, 0)\,. \end{aligned}$$

For $m \in \mathbb{N}$, $m > 0$, we define $\lambda_{m,\beta} : \{1, \dots, 4m\} \to \mathbb{R}^2$ by

$$\lambda_{m,\beta}(l) = \begin{cases} e^\beta_{\text{right}} & \text{if } l \in \{1, 2m+1\} \\ e^\beta_{\text{up}} & \text{if } 1 < l < 2m+1 \\ e^\beta_{\text{down}} & \text{if } 2m+1 < l \leq 4m \end{cases}\,. \tag{22}$$

The curve one obtains by connecting successively $\sum_{i=1}^l \lambda_{m,\beta}(i)$ for $l = 0, \dots, 4m$ is depicted in Figure 4. Note that this curve has length $4m$. Our next aim is to concatenate several curves as in Figure 4, letting m increase from 1 to some maximal value M, and then let it decrease to 1 again. For $M \in \mathbb{N}$, $M > 0$, we set

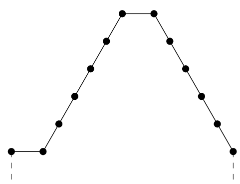

Figure 4. The curve constructed from $l \mapsto \sum_{i=1}^l \lambda_{m,\beta}(i)$ for $m = 3$.

$$\begin{aligned} a^{(M)} =& (1, 2, \dots, M-1, M, M-1, \dots, 2, 1) \\ b^{(M)}_l =& 4\sum_{j=1}^l a^{(M)}_j \qquad \text{for } l = 0, \dots, 2M-1. \end{aligned}$$

Then for $i = 1, \dots, 4M^2$, there exists a unique $l_i \in \{1, \dots, 2M-1\}$ such that

$$b^{(M)}_{l_i - 1} < i \leq b^{(M)}_{l_i}\,.$$

We set

$$\kappa_{M,\beta}(i) := \lambda_{l_i,\beta}(i - b^{(M)}_{l_i})\,.$$

Furthermore, we set $\kappa_{M,\beta}(4M^2+1) = e^\beta_{\text{right}}$, and for $j = 0, \dots, 4M^2+1$, we set

$$\mathrm{K}_{M,\beta}(j) := \sum_{i=1}^j \kappa_{M,\beta}(i)\,.$$

The curve one obtains by connecting successively $\mathrm{K}_{M,\beta}(j)$ for $j = 0, \dots, 4M^2+1$ is depicted in Figure 5. We may compute

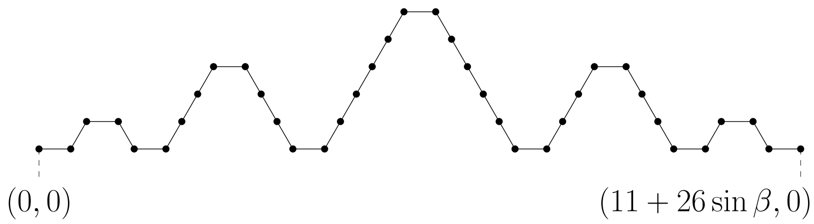

FIGURE 5. The curve constructed from $\mathrm{K}_{3,\beta}$.

$$\begin{aligned} e_1 \cdot \mathrm{K}_{M,\beta}(4M^2+1) = & \sum_{m=1}^{M} \sum_{i=1}^{4m} e_1 \cdot \lambda_{m,\beta}(i) + \sum_{m=M+1}^{2M-1} \sum_{i=1}^{4(2M-m)} e_1 \cdot \lambda_{m,\beta}(i) \\ =& 2 \left(\sum_{m=1}^{M-1} ((4m-2)\sin\beta + 2) \right) + (4M-2)\sin\beta + 3 \\ =& (4M(M-1)+2)\sin\beta + 4M - 1 \,. \end{aligned} \tag{23}$$

Now set

$$\begin{aligned} \hat{N}(M) :=& 4M^2 + 1 \\ \hat{r}(M,\beta) :=& \left(e_1 \cdot \mathrm{K}_{M,\beta}(4M^2+1)\right)^{-1} \\ =& ((4M(M-1)+2)\sin\beta + 4M - 1)^{-1} \\ \hat{d}(m,\beta) :=& -\frac{\log \hat{N}(M)}{\log \hat{r}(M,\beta)} \,. \end{aligned} \tag{24}$$

We claim that it is possible to choose $M_0 \in \mathbb{N}$ such that

$$\begin{aligned} \bar{d} <& \hat{d}(M_0, 0) \,, \\ (4M_0 - 1)^{-1} <& \frac{1}{2} \,, \\ 2(4M_0-1)^{\alpha-1} \leq& 1 \,. \end{aligned} \tag{25}$$

Indeed, we have $\hat{r}(M,0) = (4M-1)^{-1}$ and hence

$$\hat{d}(M,0) = \frac{\log(4M^2+1)}{\log(4M-1)} \,. \tag{26}$$

In particular, note that

$$\hat{d}(M,0) \to 2 \text{ as } M \to \infty \,.$$

This proves that we may choose M_0 such that the first inequality in (25) is fulfilled. After possibly increasing M_0, the second and third hold true too.

Since $\hat{r}(M_0,\cdot)$ is continuous monotone decreasing on $[0,\pi/2]$, the function $\hat{d}(M_0,\cdot)$ is continuous monotone decreasing on $[0,\pi/2]$ too. Additionally, we have $\hat{d}(M_0,\pi/2)=1$. Hence, there exists $\beta_0\in(0,\pi/2)$ such that

$$\bar{d}=\hat{d}(M_0,\beta_0)\,.$$

Now set

$$r:=\hat{r}(M_0,\beta_0)\,,\quad N:=\hat{N}(M_0).$$

We use the following notation: To two points $x\neq y\in\mathbb{R}^2$, we associate the (unique) orientation preserving Euclidean motion $S_{x,y}:\mathbb{R}^2\to\mathbb{R}^2$ that maps $(0,0)$ to x and $(1,0)$ to y.

For $i=0,\dots,N$ write $p(i):=r\mathrm{K}_{M_0,\beta_0}(i)$, and set

$$S_i:=S_{p(i-1),p(i)}\,.$$

It remains to verify that r,N and $\mathcal{S}=\{S_1,\dots,S_N\}$ satisfy the required properties.

Property (i) follows from $p(i)-p(i-1)\in\{re^{\beta_0}_{\text{right}},re^{\beta_0}_{\text{up}},re^{\beta_0}_{\text{down}}\}$ for $i=1,\dots,N$, and $|e^{\beta_0}_{\text{right}}|=|e^{\beta_0}_{\text{up}}|=|e^{\beta_0}_{\text{down}}|=1$. Property (ii) simply follows from $S_1(0,0)=0$, $S_i(1,0)=S_{i+1}(0,0)$ for $i=1,\dots,N-1$ and $S_N(1,0)=(1,0)$. Property (iii) follows from the definition of $\hat{d}(M,\beta)$ in the last line of (24), $\bar{d}=\hat{d}(M_0,\beta_0)$, $N=\hat{N}(M_0)$ and $r=\hat{r}(M_0,\beta_0)$. Property (iv) is obvious from an inspection of Figure 6, where the union of all $S_i(D)$, $i=1,\dots,N$ is depicted. Property (v) follows from the second and third line in (25), and

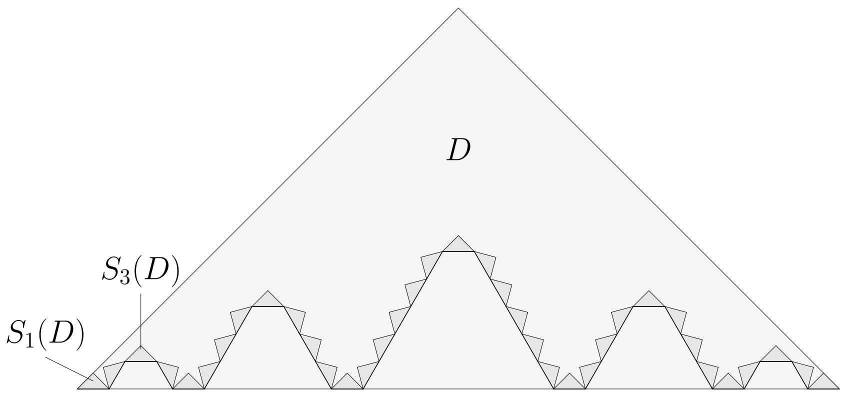

FIGURE 6. $S_i(D)\subset D$ for $i=1,\dots,N$, $S_i(D)\cap S_j(D)=\emptyset$ for $i\neq j$, and $S_i(D)\cap S_j(D)=\emptyset$ for $|i-j|>1$. In the picture above, $N=37$.

$r=\hat{r}(M_0,\beta_0)\leq(4M_0-1)^{-1}$. This concludes the proof of the lemma. □

Proof of Lemma 4.4. We are going to assume that $n=2$ and construct the maps P^m_{m+1} and P^m for this case only. The general case follows easily by setting

$$\begin{aligned}P^m_{m+1}(x)=&(P^{m,(2)}_{m+1}(x_1,x_2),x_3,\dots)\,,\\ P^m(x)=&(P^{m,(2)}(x_1,x_2),x_3,\dots)\,,\end{aligned}$$

where $P_{m+1}^{m,(2)}$, $P^{m,(2)}$ denote the maps constructed for $n = 2$ below.
Let $A \subset \overline{D}$ be the bounded closed simply connected set whose boundary contains the union of the curves L and $S(L)$, see Figure 7. I.e., the set A satisfies

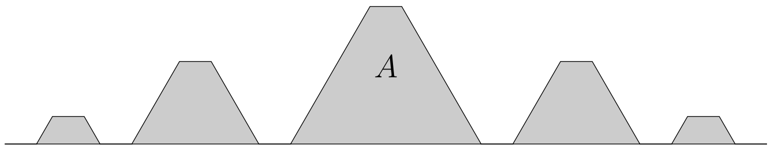

FIGURE 7. The closed set A, whose boundary contains L and $S(L)$ (with $N = 37$).

$$\overline{U^1} \setminus U^0 = \bigcup_{i=1}^{4} S_i^* A\,.$$

For $x = (x_1, x_2) \in A$, let P be the projection $A \to L$, $x \mapsto (x_1, 0)$. Obviously, P is Lipschitz with Lipschitz constant ≤ 1.
For $m \in \mathbb{N}$, $y \in \overline{U}^{m+1} \setminus U^m$, there exist $i_0 \in \{1, \dots, 4\}$, $i_1, \dots, i_m \in \{1, \dots, N\}$, and $x \in A$ such that

$$y = S_{i_0|i_1,\dots,i_m} x\,.$$

We set

$$P_{m+1}^m(y) = S_{i_0|i_1,\dots,i_m} P(x)\,.$$

We claim that

$$\begin{aligned} P_{m+1}^m :& \overline{U^{m+1}} \setminus U^m \to \partial U^m \qquad \text{is well defined,} \\ \operatorname{Lip} P_{m+1}^m \leq& 1\,. \end{aligned} \tag{27}$$

Indeed, if there exist $i_0, i_0' \in \{1, \dots, 4\}$, $i_1, \dots, i_m, i_1', \dots, i_m' \in \{1, \dots, N\}$, and $x, x' \in A$ with $y = S_{i_0|i_1,\dots,i_m} x = S_{i_0'|i_1',\dots,i_m'} x'$, then either $i_j = i_j'$ for $j = 0, \dots, m$ and $x = x'$ or $y \in \partial U^m$ and $x, x' \in L$, in which case $P(x) = x$, $P(x') = x'$ and hence $S_{i_0|i_1,\dots,i_m} P(x) = S_{i_0'|i_1',\dots,i_m'} P(x')$. This proves the first part of (27), the second part follows from $\operatorname{Lip} P \leq 1$.
For $l > m \geq 0$, let $P_l^m : \overline{U^l} \to \partial U^m$ be defined by

$$P_l^m = P_{m+1}^m \circ \dots \circ P_l^{l-1}\,.$$

It is easily seen from this definition and (27) that $\operatorname{Lip} P_l^m \leq 1$.
We come to the definition of $P^m : \overline{U} \to \overline{U^m}$ for $m \in \mathbb{N}$. We set $A^m := \overline{U^{m+1}} \setminus U^m$ and note

$$U \subset U^m \cup \left(\bigcup_{k=m}^{\infty} A^k \right)\,.$$

For $k > m$ and $x \in A^k$, we let $P^m(x) = P_k^m(x)$. Note that this makes $P^m|_U$ well defined with

$$\operatorname{Lip} P^m|_U \leq 1\,, \tag{28}$$

since for $k \neq k'$ with $x \in A^k \cap A^{k'}$, we have $P_{k'}^m(x) = P_k^m(x)$.
Now let $x \in \partial U$. There exist $i_0 \in \{1, \dots, 4\}$ and a (possibly non-unique) sequence $i_k \in$

$\{1,\dots,N\}$, $k=1,2,\dots$, such that $x\in S_{i_0|i_1,\dots,i_k}(\overline{D})$ for every $k\in\mathbb{N}$ (cf. [9], Chapter 9). Note that for $k'>k>m$,

$$P^m_{k'}\left(S_{i_0|i_1,\dots,i_{k'}}(\overline{D})\right)\subset P^m_k\left(S_{i_0|i_1,\dots,i_k}(\overline{D})\right)\,.$$

Thus there exists a unique $x'\in S^*_{i_0}L$ (that does not depend on the choice of the sequence i_k) such that

$$x'\in\bigcap_{k=m+1}^\infty P^m_k(S_{i_0|i_1,\dots,i_k}(\overline{D}))\,.$$

We set $P^m(x)=x'$.
It remains to show that P^m is Lipschitz on $\overline{U}$ with $\operatorname{Lip}P^m\le 1$. By (28), it is sufficient to show continuity on $\overline{U}$. Assume we are given x_j, $j\in\mathbb{N}$, with $x_j\in U$, $x_j\to x\in\partial U$. We need to show $P^m(x_j)\to P^m(x)$. Indeed, we will show that for every subsequence, there exists a further subsequence such that convergence holds. For any subsequence, there has to exist a further subsequence x_j (no relabeling), a sequence $i_1,i_2,\dots\in\{1,\dots,N\}$, and a monotonous increasing function $K:\mathbb{N}\to\mathbb{N}$ with $\lim_{j\to\infty}K(j)=\infty$ such that

$$x_j\in S_{i_0|i_1,i_2,\dots,i_{K(j)}}(\overline{D})\,. \tag{29}$$

In this case, we must have

$$x\in\bigcap_{k=1}^\infty S_{i_0|i_1,i_2,\dots,i_k}(\overline{D})\,,$$

and hence

$$P^m(x)\in\bigcap_{k=m+1}^\infty P^m_k\left(S_{i_0|i_1,i_2,\dots,i_k}(\overline{D})\right)\,. \tag{30}$$

By (29) and (30), we get $P^m(x_j)\to P^m(x)$. This proves the lemma. □

Proof of Lemma 4.5. For $x=(x_1,x_2,\dots,x_{n-1},0)\in B$, we will identify x with $(x_1,\dots,x_{n-1})$, and we write $\hat{x}=x/|x|$. We define $\tilde{\zeta}$ by

$$\tilde{\zeta}(x)=(\hat{x}\sin\pi|x|,\cos\pi|x|)\,.$$

For $x\in B$, we compute

$$\begin{aligned}D\tilde{\zeta}(x)=&(e_1\otimes e_1+\dots+e_{n-1}\otimes e_{n-1})\frac{\sin\pi|x|}{|x|}+\hat{x}\otimes\hat{x}\left(\pi\cos\pi|x|-\frac{\sin\pi|x|}{|x|}\right)\\&-\pi\sin\pi|x|\,e_n\otimes\hat{x}\,.\end{aligned} \tag{31}$$

From this formula, we see that $\tilde{\zeta}$ is indeed Lipschitz. All other properties claimed in (i) are verified easily.
For the proof of (ii), first note that $\zeta^{(W)}:\partial W\to\mathbf{S}^{n-1}$ is a well defined Lipschitz map. Furthermore, $\#(\zeta^{(W)})^{-1}(y)=1$ for all $y\in\mathbf{S}^{n-1}\setminus\{-e_n\}$. This implies that there exists $k\in\{-1,+1\}$ such that

$$\deg^\partial(\zeta^{(W)},\partial W,y)=\begin{cases}k & \text{for all } y\in B(0,1)\\ 0 & \text{for all } y\in\mathbb{R}^n\setminus\overline{B(0,1)}\,.\end{cases} \tag{32}$$

Next, we construct[1] a Lipschitz map $\lambda : \overline{W} \to \mathbb{R}^n$ such that

$$\lambda = \zeta \quad \text{on } \partial W, \quad \partial_n \lambda(x) = \zeta^{(W)}(x) \quad \text{for all } x \in Q\,.$$

Such a λ exists by (a suitable version of) the Whitney Extension Theorem (see, e.g., Theorem 3.6.2 in [28]). We are going to compute explicitly the sign of

$$\int_{\mathbb{R}^n} \deg^\partial(\zeta^{(W)}, \partial W, \cdot)\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^n} \deg(\lambda, W, \cdot)\mathrm{d}\mathcal{L}^n\,,$$

to decide which value for k holds true in (32). In order to do so, we introduce the following piece of notation. Let

$$\varepsilon_{i_1\dots i_n} = \begin{cases} 0 & \text{if } \{i_1,\dots,i_n\} \neq \{1,\dots,n\} \\ \operatorname{sgn}\left((1,\dots,n) \mapsto (i_1,\dots,i_n)\right) & \text{else.} \end{cases}$$

In the second line on the right hand side above, $\operatorname{sgn}\left((1,\dots,n) \mapsto (i_1,\dots,i_n)\right)$ denotes the signature of the permutation $(1,\dots,n) \mapsto (i_1,\dots,i_n)$. With this notation, we have for $x \in \overline{W}$,

$$\begin{aligned} \det D\lambda(x) &= \sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1\dots i_n}(\partial_{x_{i_1}}\lambda_1)\dots(\partial_{x_{i_n}}\lambda_n) \\ &= \sum_{i_1,\dots,i_n=1}^{n} \partial_{x_{i_n}}\left(\varepsilon_{i_1\dots i_n}(\partial_{x_{i_1}}\lambda_1)\dots(\partial_{x_{i_{n-1}}}\lambda_{n-1})\lambda_n\right) \\ &=: \sum_{i_n=1}^{n} \partial_{x_{i_n}} f_{i_n} \\ &= \operatorname{div} f\,. \end{aligned}$$

Using the formula (5) for the computation of the degree, and the Gauss-Green Theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \deg(\lambda, W, \cdot)\mathrm{d}\mathcal{L}^n &= \int_W \det D\lambda(x)\mathrm{d}x \\ &= \int_{\partial W} \nu_W \cdot f\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_B e_n \cdot f\mathrm{d}\mathcal{H}^{n-1}\,, \end{aligned}$$

where we denoted the outward normal to W by ν_W, and used the fact that f vanishes $\mathcal{H}^{n-1}$ almost everywhere on $\partial W \setminus B$. For $x \in B$, we have

$$\begin{aligned} f_n(x) &= \sum_{i_1,\dots,i_{n-1}=1}^{n} \varepsilon_{i_1\dots i_{n-1}n}\lambda_{1,i_1}(x)\dots\lambda_{n-1,i_{n-1}}(x)\lambda_n(x) \\ &= \cos\pi|x| \det D(x \mapsto \hat{x}\sin\pi|x|) \\ &= \pi\cos^2\pi|x| \left(\frac{\sin\pi|x|}{|x|}\right)^{n-2}\,, \end{aligned}$$

[1]The construction of λ is not necessary if we use the relation (6) – we do not do so here for the sake of clarity.

where the value of $(\sin \pi|x|)/|x|$ at 0 is understood to be π. It follows that

$$\int_{\mathbb{R}^n} \deg^\partial(\zeta^{(W)}, \partial W, \cdot) \mathrm{d}\mathcal{L}^n > 0\,,$$

and hence it follows from (32) that[2]

$$\deg^\partial(\zeta^{(W)}, \partial W, \cdot) \doteq \chi_{B(0,1)}\,.$$

This proves the lemma. □

Proof of Lemma 4.8. For $m \in \mathbb{N}$, choose $k_m \in \mathbb{N}$ such that

$$k_m^{-1} \geq r^m \geq (k_m + 1)^{-1}\,.$$

This choice implies

$$k_m r^m \to 1 \quad \text{as } m \to \infty\,. \tag{33}$$

Further, let $\tilde{\mathcal{Q}}^m$ denote the set of cubes in $\mathbb{R}^{n-2}$ of side length r^m and vertices in $([0,1] \cap (\mathbb{N}r^m))^{n-2}$,

$$\begin{aligned}\tilde{\mathcal{Q}}^m :=& \{(j_1 r^m, (j_1+1)r^m) \times \cdots \times (j_{n-2}r^m, (j_{n-2}+1)r^m) :\\ & 0 \leq j_l \leq k_m - 1 \text{ for } l = 1, \ldots, n-2\}\end{aligned}$$

Now let $L' = (0,1) \times \{0\}$, recall the definition of the similarities $S_{i_0|i_1,\ldots,i_m}$ from Section 2.2, and set

$$\begin{aligned}\mathcal{Q}^m := \big\{ & S_{i_0|i_1,\ldots,i_m}(L') \times \tilde{Q} : i_0 \in \{1,\ldots,4\},\\ & i_1,\ldots,i_m \in \{1,\ldots,N^m\}\,, \tilde{Q} \in \tilde{\mathcal{Q}}^m \big\}\,.\end{aligned}$$

Property (i) follows directly from the definition of ∂U^m (see Definition 4.2) and $\mathcal{Q}^m$. The first part of property (ii) is also obvious; we show the second part. Assume $Q_1, Q_2 \in \mathcal{Q}^m$, $Q_1 = S_{i_0|i_1,\ldots,i_m}(L') \times \tilde{Q}_1$, $Q_2 = S_{j_0|j_1,\ldots,j_m}(L') \times \tilde{Q}_2$, with $i_0, j_0 \in \{1,\ldots,4\}$, $i_1,\ldots,i_m, j_1,\ldots,j_m \in \{1,\ldots,N\}$, and $\tilde{Q}_1, \tilde{Q}_2 \in \tilde{\mathcal{Q}}^m$. We must have either $\overline{\tilde{Q}_1} \cap \overline{\tilde{Q}_2} = \emptyset$ or $S_{i_0|i_1,\ldots,i_m}(L) \cap S_{j_0|j_1,\ldots,j_m}(L) = \emptyset$. In the first case, $\mathrm{dist}(\tilde{Q}_1, \tilde{Q}_2) \geq r^m$ and hence also $\mathrm{dist}(Q_1, Q_2) \geq r^m$. In the second case, $\mathrm{dist}(S_{i_0|i_1,\ldots,i_m}(L), S_{j_0|j_1,\ldots,j_m}(L)) \geq r^m$ and hence also $\mathrm{dist}(Q_1, Q_2) \geq r^m$.
Next, note that by construction, $\#\mathcal{Q}^m = (\#\tilde{\mathcal{Q}}^m)N^m = (k_m)^{(n-2)}N^m$ and hence property (iii) follows from (33).

□

Appendix A. Properties of Hölder functions

The following lemma is based on the construction from the well known Whitney extension Theorem. The proof is mainly a repetition of the proof of Theorem 2.1 in [13]. However, we could not find a full proof of the claim we need in the literature, which is why we give it here.

Lemma A.1. *Let $K \subset \mathbb{R}^n$ be compact, $0 < \alpha < 1$, and $f \in C^{0,\alpha}(K)$. Then there exists $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\|\tilde{f}\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C(n,\alpha)\|f\|_{C^{0,\alpha}(K)}\,,$$

and $\tilde{f}|_K = f$.

[2] Of course, instead of arguing that $k \in \{-1, +1\}$ in (32) as we did above that equation, the value of k could also have been deduced by explicit calculation of $\int_B e_n \cdot f \mathrm{d}\mathcal{H}^{n-1}$.

Proof. Let W be the Whitney decomposition of $\mathbb{R}^n \setminus K$, see Lemma 2.5. For $Q_i \in W$, denote the sidelength of Q_i by $|Q_i|$, and its center by x_i. For each $i \in \mathbb{N}$, fix $p_i \in K$ such that $\operatorname{dist}(Q_i, p_i) = \operatorname{dist}(Q_i, K)$, and let $\tilde{Q}_i$ be the cube with center x_i and $|\tilde{Q}_i| = \frac{3}{2}|Q_i|$. Fix $\eta \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \eta \le 1$, $\eta = 1$ on $[-1,1]^n$, $\eta = 0$ on $\mathbb{R}^n \setminus [-3/2, 3/2]$ and $|D\eta| \le 4$. Set

$$
\begin{aligned}
\eta_i(x) :=& \eta\left(\frac{x - x_i}{|Q_i|}\right), \\
\varphi_i(x) :=& \frac{\eta_i(x)}{\sum_{j \in \mathbb{N}} \varphi_j(x)}.
\end{aligned}
$$

Note that $\{\varphi_i\}_{i \in \mathbb{N}}$ is a partition of unity of $\mathbb{R}^n \setminus K$ subordinate to $\tilde{W} := \{\tilde{Q}_i : i \in \mathbb{N}\}$. We define the extension $\tilde{f}$ by

$$
\tilde{f}(x) := \begin{cases} \sum_{i \in \mathbb{N}} \varphi_i(x) f(p_i) & \text{for } x \in \mathbb{R}^n \setminus K \\ f(x) & \text{for } x \in K. \end{cases}
$$

From this definition, we immediately get

$$
\sup_{\mathbb{R}^n} |\tilde{f}| \le \|f\|_{C^0(\mathbb{R}^n)}. \tag{34}
$$

Obviously, on $\mathbb{R}^n \setminus K$, $\tilde{f}$ is a smooth function, and there exists a number $N = N(n)$ such that for each $x \in \mathbb{R}^n \setminus K$, there exist at most N pairwise disjoint $Q_i \in W$ such that $\varphi_i(x) \neq 0$.

Fix $x \in \mathbb{R}^n \setminus K$. Let $\mathcal{N}(x) \subset \mathbb{N}$ denote the index set defined by $\varphi_i(x) \neq 0$ for $i \in \mathcal{N}(x)$, and let $i_0 \in \mathcal{N}(x)$. We compute

$$
\begin{aligned}
|D\tilde{f}(x)| =& \left| \sum_{i \in \mathcal{N}(x)} D\varphi_i(x) f(p_i) \right| \\
=& \left| \sum_{i \in \mathcal{N}(x) \setminus \{i_0\}} D\varphi_i(x)(f(p_i) - f(p_{i_0})) \right| \\
\le& C(n) \sum_{i \in \mathcal{N}(x)} \frac{[f]_\alpha |p_i - p_{i_0}|^\alpha}{\operatorname{dist}(Q_i, K)} \\
\le& C(n)[f]_\alpha \operatorname{dist}(x, K)^{\alpha - 1}.
\end{aligned} \tag{35}
$$

With these preparations, we are ready to prove an estimate on $[\tilde{f}]_\alpha$. Let $x, y \in \mathbb{R}^n$, and let $x', y' \in \mathbb{R}^n$ with

$$
|x - x'| = \operatorname{dist}(x, K), \quad |y - y'| = \operatorname{dist}(y, K).
$$

Note that this choice implies that for every point z on the line segment $[x, x']$, we have $|D\tilde{f}(z)| \le C[f]_\alpha |z - x'|^{\alpha - 1}$ by (35), and an analogous statement for the line segment $[y, y']$.

Hence

$$\begin{aligned}|\tilde f(x)-\tilde f(x')| =&\left|\int_0^{|x-x'|}\frac{\mathrm{d}}{\mathrm{d}t}\tilde f\left(x+t\frac{x-x'}{|x-x'|}\right)\mathrm{d}t\right|\\ \leq&\int_0^{|x-x'|}\left|D\tilde f\left(x+t\frac{x-x'}{|x-x'|}\right)\right|\mathrm{d}t\\ \leq& C(n)[f]_\alpha\int_0^{|x-x'|}t^{\alpha-1}\mathrm{d}t\\ \leq& C(n,\alpha)[f]_\alpha|x-x'|^\alpha\,.\end{aligned}$$

In the same way, we obtain $|\tilde f(y)-\tilde f(y')|\leq C[f]_\alpha|y-y'|^\alpha$.
Now assume that $|x-x'|+|y-y'|\leq 4|x-y|$. Then

$$\begin{aligned}|f(x)-f(y)| \leq&|\tilde f(x)-\tilde f(x')|+|\tilde f(x')-\tilde f(y')|+|\tilde f(y')-\tilde f(y)|\\ \leq& C(n,\alpha)[f]_\alpha(|x-x'|^\alpha+|y-y'|^\alpha+|x'-y'|^\alpha)\\ \leq& C(n,\alpha)[f]_\alpha|x-y|^\alpha\,.\end{aligned}$$

On the other hand, if $|x-x'|+|y-y'|>4|x-y|$, then we may assume $|x-x'|>2|x-y|$ and hence the line segment $[x,y]$ is contained in $\mathbb{R}^n\setminus K$, and for each point $z\in[x,y]$, we have $|D\tilde f(z)|\leq C[f]_\alpha|x-y|^{\alpha-1}$. Then we get

$$\begin{aligned}|\tilde f(x)-\tilde f(y)| =&\left|\int_0^{|x-y|}\frac{\mathrm{d}}{\mathrm{d}t}\tilde f\left(x+t\frac{x-y}{|x-y|}\right)\mathrm{d}t\right|\\ \leq&\int_0^{|x-y|}C(n)[f]_\alpha|x-y|^{\alpha-1}\mathrm{d}t\\ \leq& C(n,\alpha)[f]_\alpha|x-y|^\alpha\,.\end{aligned}$$

This proves $[\tilde f]_\alpha\leq C(n,\alpha)[f]_\alpha$, and together with (34), this proves the claim of the present lemma. □

The following lemma is a well known fact from real interpolation theory; we include it for the non-specialist reader.

Lemma A.2. *Let $U\subset\mathbb{R}^n$ be open and bounded, $u\in C^{0,\alpha}(U)$, and $v:\mathbb{R}^+\to C^1(U)$ with $v'(t)\in C^0(U)$ for all $t>0$ such that*

- $\|t^{1-\alpha}v_i(t)\|_{L^\infty(\mathbb{R}^+;C^1(U))}\leq C\|u\|_{C^{0,\alpha}(U)}$
- $\|t^{1-\alpha}v_i'(t)\|_{L^\infty(\mathbb{R}^+;C^0(U))}\leq C\|u\|_{C^{0,\alpha}(U)}$
- $\lim_{t\to 0}v_i(t)=u_i$ *in* $C^0(U)$ *.*

Then

$$\|v(t)\|_{C^{0,\alpha}(U)}\leq C\|u\|_{C^{0,\alpha}(U)}\text{ for }t\leq 1\,.$$

Proof. We first observe for $x\in\mathbb{R}^n$, and $t\leq 1$,

$$\begin{aligned}|v(t)(x)| \leq&\left|u(x)+\int_0^t v'(s)(x)\mathrm{d}s\right|\\ \leq&\|u\|_{C^0(U)}+\int_0^t C\|u\|_{C^{0,\alpha}(U)}s^{\alpha-1}\mathrm{d}s\\ \leq& C\|u\|_{C^{0,\alpha}(U)}\,.\end{aligned}$$

Now let $x, y \in \mathbb{R}^n$ with $x \neq y$. First assume $|x-y| \leq t$. Then

$$\begin{aligned}|v(t)(x) - v(t)(y)| \leq& |x-y| \sup_{\mathbb{R}^n} |Dv(t)| \leq C\|u\|_{C^{0,\alpha}(U)} |x-y| t^{\alpha-1} \\ \leq& C\|u\|_{C^{0,\alpha}(U)} |x-y|^{\alpha} .\end{aligned}$$

On the other hand, if $|x-y| > t$, then

$$\begin{aligned}|v(t)(x) - v(t)(y)| \leq& |v(t)(x) - u(x)| + |u(x) - u(y)| + |u(y) - v(t)(y)| \\ \leq& C\|u\|_{C^{0,\alpha}(U)} \left(2t^{\alpha} + |x-y|^{\alpha}\right) \\ \leq& C\|u\|_{C^{0,\alpha}(U)} |x-y|^{\alpha} .\end{aligned}$$

This proves $[v(t)]_{\alpha} \leq C\|u\|_{C^{0,\alpha}(U)}$ and hence the claim of the lemma. □

References

[1] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/77.

[2] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[3] J. Bourgain, H. Brezis, and P. Mironescu. On the structure of the Sobolev space $H^{1/2}$ with values into the circle. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(2):119–124, 2000.

[4] J. Bourgain, H. Brezis, and P. Mironescu. $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation. *Publ. Math. Inst. Hautes Études Sci.*, 99:1–115, 2004.

[5] H. Brezis and H.-M. Nguyen. The Jacobian determinant revisited. *Invent. Math.*, 185(1):17–54, 2011.

[6] H. Brezis and L. Nirenberg. Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.)*, 1(2):197–263, 1995.

[7] H. Brezis and L. Nirenberg. Degree theory and BMO. II. Compact manifolds with boundaries. *Selecta Math. (N.S.)*, 2(3):309–368, 1996. With an appendix by the authors and Petru Mironescu.

[8] S. Conti, C. De Lellis, and L. Székelyhidi Jr. h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings. In *Nonlinear Partial Differential Equations*, pages 83–116. Springer, 2012.

[9] K. Falconer. *Fractal geometry.* John Wiley & Sons, Ltd., Chichester, third edition, 2014. Mathematical foundations and applications.

[10] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.

[11] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[12] I. Fonseca and W. Gangbo. *Degree theory in analysis and applications*, volume 2 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[13] J. Harrison and A. Norton. Geometric integration on fractal curves in the plane. *Indiana Univ. Math. J.*, 40(2):567–594, 1991.

[14] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[15] T. Iwaniec. On the concept of the weak Jacobian and Hessian. In *Papers on analysis*, volume 83 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pages 181–205. Univ. Jyväskylä, Jyväskylä, 2001.

[16] J. L. Lions. Sur les espaces d'interpolation; dualité. *Math. Scand.*, 9:147–177, 1961.

[17] J.-L. Lions and J. Peetre. Propriétés d'espaces d'interpolation. *C. R. Acad. Sci. Paris*, 253:1747–1749, 1961.

[18] J.-L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.*, 19:5–68, 1964.

[19] J. Malý and O. Martio. Lusin's condition (N) and mappings of the class $W^{1,n}$. *J. Reine Angew. Math.*, 458:19–36, 1995.

[20] M. Marcus and V. J. Mizel. Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. *Bull. Amer. Math. Soc.*, 79:790–795, 1973.

[21] O. Martio and M. Vuorinen. Whitney cubes, p-capacity, and Minkowski content. *Exposition. Math.*, 5(1):17–40, 1987.
[22] O. Martio and W. P. Ziemer. Lusin's condition (N) and mappings with nonnegative Jacobians. *Michigan Math. J.*, 39(3):495–508, 1992.
[23] Charles B. Morrey, Jr. *Multiple integrals in the calculus of variations*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1966 edition [MR0202511].
[24] S. Müller. Det = det. A remark on the distributional determinant. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(1):13–17, 1990.
[25] J. G. Rešetnjak. The weak convergence of completely additive vector-valued set functions. *Sibirsk. Mat. Z*, 9:1386–1394, 1968.
[26] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
[27] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin; UMI, Bologna, 2007.
[28] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.
[29] R. Züst. Integration of Hölder forms and currents in snowflake spaces. *Calc. Var. Partial Differential Equations*, 40(1-2):99–124, 2011.

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